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HANKEL DETERMINANT FOR STARLIKE AND CONVEX FUNCTIONS OF ORDER $\frac{\alpha}{2}$

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ABSTRACT

The aim of this paper is to obtain an upper bound to the second Hankel Determinant $\left|a_2a_4-a_3^2\right|$ for starlike and convex functions of order $\frac{\alpha}{2},(1\le\alpha\le 2)$

Key words: Analytic function, second Hankel functional, starlike and convex functions, upper bound.

1. INTRODUCTION

Let A denote the class of functions f of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad (1.1)$$

in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$.

The Hankel determinant of f for $q\ge 1$ and $n\ge 1$ was defined by Pommerenke ([6], [7]) as

$$H_q(\mathrm{n})=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (1.2)$$

This determinant has been considered by many authors in the literature [8,21,22]. For example

Noor [9] determined the rate of growth of $H_q(\mathrm{n})$ as $n\to\infty$ for functions given by (1.1)
with bounded boundary. Ehrenborg [10] studied the Hankel determinant of exponential
polynomials. Janteng et al. discussed the Hankel determinant problem for the classes of
starlike functions with respect to symmetric points and convex functions with respect to
symmetric points in [11] and for the functions whose derivative has a positive real part in
[12].

Easily, one can observe that the Fekete and Szego functional is $H_2(1)$. Fekete and Szego
[13] then further generalised the estimate $\left|a_3-\mu a_2^2\right|$, where μ is real and $f\in S$. For our
discussion in this paper, we consider the Hankel determinant in the case of $q=2$ and $n=2$:

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Let P denote the class of functions

$$P(\mathrm{z})=1+\mathrm{c}_1z+c_2z^2+c_3z^3+\cdots=1+\sum_{n=1}^{\infty}c_nz^n, \qquad (1.3)$$

which are analytic in U and satisfy $\operatorname{Re}\{P(\mathrm{z})\}>0$ for any $z\in U$.

In this paper, we seek sharp upper bound to the functional $\left|a_2a_4-a_3^2\right|$ for the function
f belonging to the class S_α^* and C_α. The class S_α^* and C_α. are defined as follows.

Definition 1.1. Let f be given by (1.1). Then $f \in S_{\alpha}^{*}$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{\alpha}{2}-1, (1 \le \alpha \le 2). \quad (1.4)$$

Definition 1.1. Let f be given by (1.1). Then $f \in C_{\alpha}$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{\alpha}{2}-1, (1 \le \alpha \le 2). \quad (1.4)$$

To prove our main result in the next section, we shall require the following two Lemmas:

Lemma 1.3. ([1], [2]) If $c \in P$, then $|c_n| \le 2$, for each $n \ge 1$.

Lemma 1.4. ([3], [4], [5]) If $c \in P$, then

$$2c_2 = c_1^2 + (4-c_1^2)x,$$
$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2),$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $c_1 \in [0,2]$.

We employ techniques similar to these used earlier by Amourah et al. ([15], [16], [17], [18], [20]) and Al-Hawary et al. [19].

2. MAIN RESULT

Theorem 2.1. If $f \in S_{\alpha}^{*}$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{(2-\frac{\alpha}{2})^2}{3}\left[4\left(\left(2-\frac{\alpha}{2}\right)^2-1\right)+3\right], (1 \le \alpha \le 2).$$

(2.1)

Proof. Since $f \in S_{\alpha}^{*}$, by Definition 1.1 we have

$$\frac{1-\frac{\alpha}{2}+\frac{zf'(z)}{f(z)}}{2-\frac{\alpha}{2}} = P(z). \quad (2.2)$$

Replacing $f(z)$ and $q(z)$ with their equivalent series expressions in (2.2), we have

$$\left(1-\frac{\alpha}{2}+\frac{zf'(z)}{f(z)}\right) = \left(2-\frac{\alpha}{2}\right)P(z).$$

Using the Binomial expansion in the left hand side of the above expression, upon simplification, we obtain

$$\left(2-\frac{\alpha}{2}\right)z + \left(2-\frac{\alpha}{2}\right)(c_1+a_2)z^2 + \left[c_2+a_2c_1+a_3\right]\left(2-\frac{\alpha}{2}\right)z^3$$
$$+\left[c_3+a_2c_2+a_3c_1+a_4\right]\left(2-\frac{\alpha}{2}\right)z^4+\cdots$$
$$=\left(2-\frac{\alpha}{2}\right)z + \left(3-\frac{\alpha}{2}\right)a_2z^2 + \left(4-\frac{\alpha}{2}\right)a_3z^3+\cdots. \quad (2.3)$$

On equating coefficients in (2.3), we get

$$a_2 = \left(2-\frac{\alpha}{2}\right)c_1, a_3 = \frac{\left(2-\frac{\alpha}{2}\right)}{2}\left(c_2+\left(2-\frac{\alpha}{2}\right)c_1^2\right),$$
$$a_4 = \frac{\left(2-\frac{\alpha}{2}\right)}{6}\left(2c_3+3\left(2-\frac{\alpha}{2}\right)c_1c_2+\left(2-\frac{\alpha}{2}\right)c_1^3\right), \quad (2.4)$$

in the second Hankel functional

$$\left|a_2a_4-a_3^2\right| = \frac{\left(2-\frac{\alpha}{2}\right)^2}{12}\left|4c_1c_3-3c_2^2-c_1^4\left(2-\frac{\alpha}{2}\right)^2\right|.$$

Using Lemma 1.4, it gives

$$\left|a_2a_4-a_3^2\right| = \left(2-\frac{\alpha}{2}\right)^2\left|\frac{(4-c_1^2)c_1^2x}{24}-\frac{\left(4\left(2-\frac{\alpha}{2}\right)^2-1\right)c_1^4}{48}+\frac{(4-c_1^2)(1-|x|^2)c_1z}{6}-\frac{(4-c_1^2)x^2(12+c_1^2)}{48}\right|.$$

Assume that $\delta = |x| \le 1$, $c_1 = c$ and $c \in [0,2]$, using triangular inequality and $|z| \le 1$, we have

$$\left|a_2a_4-a_3^2\right| \le \left(2-\frac{\alpha}{2}\right)^2\left\{\frac{(4-c_1^2)c^2\delta}{24}+\frac{\left(4\left(2-\frac{\alpha}{2}\right)^2-1\right)c^4}{48}+\frac{(4-c^2)(1-\delta^2)c}{6}+\frac{(4-c^2)\delta^2(12+c^2)}{48}\right\}$$

(2.5)

$$=\frac{\left(2-\frac{\alpha}{2}\right)^2}{48}\left\{\begin{array}{l}\left(4\left(2-\frac{\alpha}{2}\right)^2-1\right)c^4+8\left(4-c^2\right)c+2\left(4-c_1^2\right)c^2\delta+\\ +(c-6)(c-2)\left(4-c^2\right)\delta^2\end{array}\right\}$$

(2.6)

$$=F(c,\delta). \quad (2.7)$$

We next maximize the function $F(c,\delta)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\delta)$ in (2.7) partially with respect to $\delta,$ we get

$$\frac{\partial F}{\partial\delta}=\frac{\left(2-\frac{\alpha}{2}\right)^2}{48}\left[2\left(4-c_1^2\right)c^2+2(c-6)(c-2)\left(4-c^2\right)\delta\right]$$

$$=\frac{\left(2-\frac{\alpha}{2}\right)^2}{24}\left[c^2+(c-6)(c-2)\delta\right]\left(4-c_1^2\right)$$

We have $\frac{\partial F}{\partial\delta}>0$. Thus $F(c,\delta)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$

$$\underset{0\le\delta\le1}{F(c,\delta)}=F(c,1)=G(c).$$

$$G(c)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{48}\left[4\left(\left(2-\frac{\alpha}{2}\right)^2-1\right)c^4+48\right]. \quad (2.8)$$

$$G^{/}(c)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{48}\left[16\left(\left(2-\frac{\alpha}{2}\right)^2-1\right)c^3\right]. \quad (2.9)$$

From the expression (2.9), we observe that $G^{/}(c)\ge0$ for all values of $0\le c\le2$ and $1\le\alpha\le2$. Therefore, $G(c)$ is a monotonically increasing function of c in the interval $[0,2]$ so that its maximum value occurs at $c=2$. From (2.8), we obtain

$$\max_{0\le c\le2}G(2)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{3}\left[4\left(\left(2-\frac{\alpha}{2}\right)^2-1\right)+3\right]. \quad (2.10)$$

From the expressions (2.7) and (2.10), we obtain

$$\left|a_2a_4-a_3^2\right|\le\frac{\left(2-\frac{\alpha}{2}\right)^2}{3}\left[4\left(\left(2-\frac{\alpha}{2}\right)^2-1\right)+3\right]. \quad (2.11)$$

This completes the proof of our theorem 2.1.

In particular, considering $\alpha=2$ in Theorem 2.1, we have the following result.

Remark 2.2. If $f\in S_{\alpha}^{*}$ then

$$\left|a_2a_4-a_3^2\right|\le1.$$

This inequality is sharp and coincides with that of Janteng, Halim and Darus [14].

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Theorem 2.1. If $f\in C_{\alpha}$, then

$$\left|a_2a_4-a_3^2\right|\le\frac{(2-\frac{\alpha}{2})^2}{144}\left[\frac{17\left(2-\frac{\alpha}{2}\right)^2+2\left(2-\frac{\alpha}{2}\right)+17}{1+\left(2-\frac{\alpha}{2}\right)^2}\right],(1\le\alpha\le2).$$

(2.12)

Proof. Since $f\in C_{\alpha}$, by Definition 1.2 we have

$$\frac{2-\frac{\alpha}{2}+\frac{zf^{//}(z)}{f^{/}(z)}}{2-\frac{\alpha}{2}}=P(z).$$

(2.2)

Replacing $f(z)$ and $q(z)$ with their equivalent series expressions in (2.13), we have

$$\left(-\frac{\alpha}{2}+\frac{zf^{//}(z)}{f^{/}(z)}\right)=\left(2-\frac{\alpha}{2}\right)P(z).$$

Using the Binomial expansion in the left hand side of the above expression, upon simplification, we obtain

$$\left(2-\frac{\alpha}{2}\right)c_1z+\left[c_2+2a_2c_1+a_3\right]\left(2-\frac{\alpha}{2}\right)z^2$$
$$+\left[c_3+2a_2c_2+3a_3c_1\right]\left(2-\frac{\alpha}{2}\right)z^3+\cdots$$
$$=2a_2z+6a_3z^2+12a_4z^3+\cdots.$$

(2.14)

On equating coefficients in (2.14), we get

$$a_2=\frac{\left(2-\frac{\alpha}{2}\right)c_1}{2},\ a_3=\frac{\left(2-\frac{\alpha}{2}\right)}{6}\left(c_2+\left(2-\frac{\alpha}{2}\right)c_1^2\right),$$

$$a_4=\frac{\left(2-\frac{\alpha}{2}\right)}{12}\left(c_3+\frac{3\left(2-\frac{\alpha}{2}\right)}{2}c_1c_2+\frac{\left(2-\frac{\alpha}{2}\right)}{2}c_1^3\right), \quad (2.15)$$

in the second Hankel functional

$$\left|a_2a_4-a_3^2\right|=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left|6c_1c_3-4c_2^2+\left(2-\frac{\alpha}{2}\right)c_1^2c_2-c_1^4\left(2-\frac{\alpha}{2}\right)^2\right|.$$

Using Lemma 1.4, it gives

$$\left|a_2a_4-a_3^2\right|=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left|\frac{\left(1+\left(2-\frac{\alpha}{2}\right)-2\left(2-\frac{\alpha}{2}\right)^2\right)c_1^4}{2}+\left(\frac{2+\left(2-\frac{\alpha}{2}\right)}{2}\right)\left(4-c_1^2\right)c_1^2x-\frac{(8+c_1^2)}{2}\left(4-c_1^2\right)x^2+3\left(4-c_1^2\right)\left(1-|x|^2\right)c_1z\right|$$

Assume that $\delta=|x|\le 1$, $c_1=c$ and $c\in[0,2]$, using triangular inequality and $|z|\le 1$, we have

$$\left|a_2a_4-a_3^2\right|\le\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left\{\frac{\left(1+\left(2-\frac{\alpha}{2}\right)-2\left(2-\frac{\alpha}{2}\right)^2\right)c^4}{2}+3c\left(4-c^2\right)+\left(\frac{2+\left(2-\frac{\alpha}{2}\right)}{2}\right)\left(4-c^2\right)c^2\delta+\frac{(c-2)(c-4)\left(4-c^2\right)}{2}\delta^2\right\} \quad (2.16)$$

$$=F(c,\delta).$$

We next maximize the function $F(c,\delta)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\delta)$ in (2.16) partially with respect to δ, we get

$$\frac{\partial F}{\partial\delta}=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left[\left(\frac{2+\left(2-\frac{\alpha}{2}\right)}{2}\right)c^2\left(4-c^2\right)+(c-4)(c-2)\left(4-c^2\right)\delta\right].$$

We have $\frac{\partial F}{\partial\delta}>0$. Thus $F(c,\delta)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$

$$\max_{0\le\delta\le 1}F(c,\delta)=F(c,1)=G(c).$$

$$G(c)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left\{\frac{\left(1+\left(2-\frac{\alpha}{2}\right)-2\left(2-\frac{\alpha}{2}\right)^2\right)c^4}{2}+3c\left(4-c^2\right)+\left(\frac{2+\left(2-\frac{\alpha}{2}\right)}{2}\right)\left(4-c^2\right)c^2+\frac{(c-2)(c-4)\left(4-c^2\right)}{2}\right\}, \quad (2.8)$$

$$=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left\{-\left[\left(2-\frac{\alpha}{2}\right)^2+1\right]c^4+2\left[\left(2-\frac{\alpha}{2}\right)+1\right]c^2+16\right\}. \quad (2.17)$$

$$G'(c)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left\{-4\left[\left(2-\frac{\alpha}{2}\right)^2+1\right]c^3+4\left[\left(2-\frac{\alpha}{2}\right)+1\right]c\right\}. \quad (2.18)$$

$$G''(c)=\frac{\left(2-\frac{\alpha}{2}\right)^2}{144}\left\{-12\left[\left(2-\frac{\alpha}{2}\right)^2+1\right]c^2+4\left[\left(2-\frac{\alpha}{2}\right)+1\right]\right\}. \quad (2.19)$$

For Optimum value of $G(c)$, consider $G'(c)=0$. From (2.18), we get

$$c\left[-\left[\left(2-\frac{\alpha}{2}\right)^2+1\right]c^2+\left[\left(2-\frac{\alpha}{2}\right)+1\right]\right]=0. \quad (2.20)$$

We now discuss the following Cases.

Case 1) If $c = 0$, then, from (2.19), we obtain

$$G^{//}(c) = \frac{\left(2-\frac{\alpha}{2}\right)^2}{36}\left[\left(2-\frac{\alpha}{2}\right) +1\right] > 0$$

From the second derivative test, $G(c)$ has minimum value at $c = 0$.

Case 2) If 6= 0, then, from (2.19), we get

$$c^2 = \frac{1+\left(2-\frac{\alpha}{2}\right)}{1+\left(2-\frac{\alpha}{2}\right)^2}. \quad (2.21)$$

Using the value of c^2 given in (2.21) in (2.19), after simplifying, we obtain

$$G^{//}(c) = \frac{-60\left(2-\frac{\alpha}{2}\right)^2}{144}\left[\left(2-\frac{\alpha}{2}\right) +1\right] < 0.$$

By the second derivative test, $G(c)$ has maximum value at c, where c^2 given in (2.21).

Using the value of c^2 given by (2.21) in (2.17), upon simplification, we obtain

$$\max_{0\le c\le 2} G(c) = \frac{(2-\frac{\alpha}{2})^2}{144}\left[\frac{17\left(2-\frac{\alpha}{2}\right)^2 + 2\left(2-\frac{\alpha}{2}\right)+17}{1+\left(2-\frac{\alpha}{2}\right)^2}\right].$$

(2.22)

Considering, the maximum value of $G(c)$ at c, where c^2 is given by (2.21) , from (2.16) and (2.22), we obtain

$$\left|a_2a_4 - a_3^2\right| \le \frac{(2-\frac{\alpha}{2})^2}{144}\left[\frac{17\left(2-\frac{\alpha}{2}\right)^2 + 2\left(2-\frac{\alpha}{2}\right)+17}{1+\left(2-\frac{\alpha}{2}\right)^2}\right].$$

(2.23)

This completes the proof of our Theorem 2.3.

Remark 2.4. If $f \in C_\alpha$,then

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8}.$$

This inequality is sharp and coincides with that of Janteng, Halim and Darus [14].

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