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General Balanced Magic Squares

Saleem Al-ashhab[1], Laith Abualigah[2]
[1] Department of Mathematics, AABU, Mafraq, Jordan
[2] Faculty of Computer Science and Informatics, Computer Science Department, Amman Arab University, Amman , Jordan

ABSTRACT

In this paper we consider the problem of counting magic squares 6 by 6. We introduce and study special types of magic squares of order six. We present the property preserving transformations. We list the enumerations of the squares, which processes special features.

Key words : *Magic squares; Four corner propery; Balanced magic squares*

1. INTRODUCTION

In this paper we consider the old famous problem of magic squares. A semi magic square is a square matrix, where the sum of all entries in each column or row yields the same number. Some authors call it magic square. This number is called the magic constant. We call a semi magic square a magic square if both main diagonals sum up to the magic constant. A natural magic square of order *n* is a matrix of size $n\times n$ such that its entries consist of all integers from one to n^2. The magic constant is in this case

$$0.5n(n^2+1).$$

a natural magic square of order three

8	1	6
3	5	7
4	9	2

The combinations, which appear in the columns, rows and both diagonals of this square, are the only distinct three element combinations of the numbers from 1 to 9 with sum 15. A symmetric magic square is a natural magic square of order *n* such that the sum of both elements of each pair of dual (opposite entries) is equal to

$$n^2+1.$$

A pandiagonal magic square is a magic square such that the sum of all entries in all broken diagonals equals the magic constant. For example, we note in table 2 that the sum of the entries 34, 36, 7, 44, 10, 2, 42 is 175, which is the magic constant. These entries represent the first right broken diagonal.

A natural pandiagonal and symmetric magic square of order seven

39	34	21	35	8	37	1
9	12	36	24	19	48	27
30	17	46	7	32	3	40
6	28	25	22	44	5	45
18	43	4	33	20	10	47
31	26	14	38	41	23	2
42	15	29	16	11	49	13

A pandiagonal and symmetric magic square is called super magic. A complete Magic square is a pandiagonal square with some supplementary qualities. For a complete Magic square of order 4, the sum of the entries: $a_{11}, a_{12}, a_{21}, a_{22}$, $a_{11}, a_{13}, a_{31}, a_{33}$ and $a_{11}, a_{14}, a_{41}, a_{44}$ is also equal to the magic sum.

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It is well known that we have only eight 3x3 magic squares (with sum in all directions 15). All these squares have the number 5 as a middle entry and all these squares can be formed using the following transformations: rotations with angles $90^\circ, 180^\circ, 270^\circ$ and reflections about the middle column, middle row and both diagonals of the square.

In the seventeenth century F. Bessy was the first person to state that the number of the 4x4 magic squares is 880, where he considered a magic square with all its rotations and reflections one square. Hire listed later these squares in tables in the year 1693. Today we can use the computer to check that there are

880*8 = 7040

magic squares of order four. At the beginning of the twentieth century these squares were classified theoretically into twelve classes. One of these classes is the class of pandiagonal magic squares consisting of 48 squares. It was proven that they are generated by three basic squares (cf. [1]). In 1973 Schoeppel

found the number of all natural magic squares of order five. He computed it using an elementary computer. It is
64 826 306 * 32 = 2 202 441 792,
where we multiply by 32 due to the existence of a property preserving transformations. According to [2] there exists
736 347 893 760
natural nested magic squares of order six. According to [3] the number of super magic squares of order five is sixteen and number of super magic squares of order seven is 20 190 684. The number of complete magic squares of order four is 48, and the number of complete magic squares of order eight (cf. [4]) is 368 640. It is well-known that there are pandiagonal magic squares and symmetric squares of order five. It was proven that the pandiagonal magic squares are generated through 144 basic squares. Hence, there are
144 * 200 = 28 800
natural pandiagonal squares of order five. But, there are neither pandiagonal magic squares nor symmetric squares of order six. The proof can be found in [3]. The number of natural magic squares of order six is actually unknown up to day. In [5] Trump obtained using empirical methods (Monte Carlo Method) the following interval estimation for this number

(1.7712 e19, 1.7796 e19)

with a probability of 99%. We give here the number of a subset of such squares. We define here classes of magic squares of order six, which satisfy some of the conditions for both types. In [4] we find an enumeration of some subsets of pandiagonal squares. In the references [5], …, [10] we find some partial listings for the number of magic squares. In [11,12] there is an enumeration of Franklin squares, which are special magic squares of order 8 by 8.

1.2 Four corner magic square

This concept was introduced by Al-Ashhab for the first time in [5]. Al-Ashhab studied this type there in some simple cases. In [6] Al-Ashhab considered the type called nested four corner magic square with a pandiagonal magic square, where the inside 4 by 4 square was pandiagonal. In [5] we find an enumeration of this class of squares. We can find other enumerations of other classes of squares of this type in the references [7], [8], [9], [10] and [6]. Alashhab and Trump computed in 2015 the number of natural four corner squares. It is 8 730 627 225 792.

1.3 Objectives and Benefits of our Work

The study was focused on squares with centres, which are symmetric, semi symmetric or have positive determinants. In the following subsection we illustrate the previous concepts. In this paper we summarize and present the total enumerations concerning four corner magic squares.

2. BASIC THEORIES

[1] 2.1. The types of squares

A four corner magic square is a magic squares of order six with magic constant 3s such that the equations

$$a_{33} + a_{34} + a_{43} + a_{44} = 2s,$$

$$a_{ii} + a_{i(i+3)} + a_{(i+3)i} + a_{(i+3)(i+3)} = 2s$$

hold for i=1,2,3. A four corner magic square of order 6 can be written as

y	f	g	t	M	G
z	h	n	j	q	N
w	E	e	a	m	D
A	k	Q	b	H	R
V	p	d	o	Z	T
B	F	I	J	L	Y

where
A = 2s – b – t – y, B = b + j + o + t – s – w,
D = d + g + n + y – a – p – q,
E = 3s – a – e – m – w – D, F = 3s– f – h – k – p – E,
G = 2s + e + w – (j + o + p + q + t),
H = e + g + s + w + y – j – k – o – p – q,
I = a + b + s – d – g – n, J = 3s – a – b – j – o – t,
M = 3s – f – g – t – y – G, N = 3s – h – j – n – q – z,
L = f + h + k + p – m – s, Q = 2s – a – b – e,
R = a + b + j + o + p + q + t – g – 2s – w,
T = h + j + q+ z – d – s,V = 2s – j – o – z
Y = p + q + s – b – e – y, Z = 2s – p – q – h.
We see that it has seventeen independent variables, which are represented by the small letters. Further, we see that
A + p + I + t + q + D = 3s,
R + Z + J + g + h + w = 3s.
That is two broken diagonals sum up to the magic constant. In this sense we can think about this new type of squares as a partial type of pandiagonal magic squares 6 by 6.
We call a four corner magic squares such that

$$a_{33}a_{44} + a_{34} + a_{43} < 0 \ (> 0) \qquad (1)$$

a four corner magic square of order six with negative (positive) center. This means that the 2 by 2 square in the center has negative (positive) determinant.

The number of all different possible values for a, b and e by computing the number of four corner magic squares is 3429. Hence, there are 3429 possible centers of the natural four corner magic squares. The number of squares with positive center is 232. Hence, there are 3197 possible centers of the negative four corner magic squares. Among these squares there are 153 (res. 306) centers of the four corner magic squares, which are symmetric (res. semi symmetric).

A natural four corner magic square

[2] 2.2. Property preserving transformations

There are seven classical transformations, which take a magic square into another magic square. These transformations also preserve the property "four corner magic". Now, a four corner magic squares can be transformed by executing the following interchanges simultaneously into another one of the same kind:

interchange a_{12} (res. a_{62}) with a_{15} (res. a_{65})

interchange a_{21} (res. a_{26}) with a_{51} (res. a_{56})

interchange a_{22} (res. a_{25}) with a_{55} (res. a_{52})

interchange a_{23} (res. a_{24}) with a_{53} (res. a_{54})

interchange a_{32} (res. a_{42}) with a_{35} (res. a_{45})

It is obvious that the center remains unchanged by this transformation. This means that a square with negative (positive) center will be transformed into another one of the same kind. We can use this transformation to reduce the number of computed natural magic squares. In order to eliminate the effect of the previous transformations we compute all natural four corner magic squares for which the following conditions hold:

$$a < 2s - a - b - e, +a < e < b, \ p < q \ (> 0) \qquad (2)$$

[3] This means that we compute first the number of all natural squares satisfying these conditions. We multiply then this number by sixteen in order to get the number of squares. **[4]**

[5] 2.3. The Semi Pandiagonal Magic Squares

We can generalize the concept of four corner magic square to the semi pandiagonal magic square. It has the following structure:

where
A=4s – 2d – f – h – l – n – p – 2q – 2a + 2u + 2v – x – y + 2z,
B=a – c + d + h + l + n + p + 2q – s – 2u – 2v + x + y – 2z,
D=o – k – l – h + s + e, E=2s – o – m – e, F= m – a + o – s + v + z,
G=2s – u – v – z,
H = 4s – l – p – r – i – k – x – y, J = s – l – d + u – x + z,
L = d – c + l + m + o + p + q – s – 2u – v + x + y – z,
M = c – a – d – h – l – m – n – o – p – 2q + 4s + 2u + v – x – y + z,
N = 3s – k – i – c – u – z,
Q=2a + 2d + f + h + l + m + n + 2q – r – 3s – 2u – 2v + x + y – 2z + e,
R=c – d + k – m – o – q + i + u + z,
T=s – q – p + u + v – y, W=k – f + l – m + p + r + i – s + x – e,
Y=3s – o – I – n – x – e.

It is easy to see that each four corner magic is a semi pandiagonal magic square. Further, the transformations considered in 2.3 are property preserving for this new type.

It is worth mentioning that the two dependent variables in the frame of center square (E and H) depends only on the variables in the outer frame. This is helpful by programming in order to reduce run time. The problem of counting the natural squares of this type of squares is yet unsolved.

6	23	11	13	33	25
19	28	36	3	7	18
2	29	1	17	27	35
21	8	22	34	10	16
32	9	15	20	30	5
31	14	26	24	4	12

a	i	c	d	f	z
h	o	o	o	o	s – h
p	o	u	v	o	R
q	o	z	G	o	T
n	o	o	o	o	s – n
m	q	t	w	j	k

[6] 3.0 Symbolic Computations of the Determinant

It is sometimes of interest to determine the determinant of the magic square as a square matrix. In the case of the semi pandiagonal magic squares there are cases when the determinant is zero. In general the determinant is not zero for any semi pandiagonal magic square. In case we have all entries of the frame of outer 4 by 4 center (E, k, l, m, r, H, p, y, o and e) equal to the value $0.5s$. Then, we can compute using symbolic calculation software that the determinant is zero. In fact we can verify that any square of the following type has this property:

A symbolic square with zero determinant

a	A	c	d	f	B
h	E	k	l	m	D
L	r	u	v	H	R
q	p	z	G	y	T
n	o	i	x	e	Y
M	Q	N	W	J	F

We used computers to count the several types of four corner magic squares. The used code can be found in [11]. The new results in this paper are the enumeration of four corner magic squares having a negative determinant center, which is neither symmetric nor semi symmetric. Moreover, the value of a is 6, 7 or 8. In other words we consider the all centers such that

$a < e < b$, $5 < a < 9$, $2s - a - b - e > a$, $a(2s - a - b - e) > be$ (3)

In the following tables we list the number for squares associated with such centres:

list of the number with a =6

b	Center	number	b	Center	number
17,…	27	346261873	29	20	272033548
22,23	22	287762795	30	20	283550183
24	12	161521602	32	22	318578923
25	14	183266068	33	22	317504330
26	15	194563395	34	20	305173556
27	17	226208507	35	20	298075640
28	17	230224612	36	18	277953702

The total number of centers associated with a = 6 is 266. The total number of the squares is
37 026 787 410.

list of the number with a =7

b	Center	number	b	Center	Number
17,…	25	316631539	29	18	247062408
22,23	19	243590512	31	20	283550183
24	12	154169309	32	20	278823415
25	12	155140031	33	18	259302122
26	15	196132602	34	18	260058312
27	15	197439548	35	16	238403631
28	18	245964322	36	16	237122233

The total number of centers associated with a = 7 is 242. The total number of the squares is

33 133 901 727.
list of the number with a =8

b	Center	number	b	Center	number
17,…	20	246525148	30	18	247062408
22	9	111325270	31	18	242346384
23	9	112204862	32	16	226856629
24	10	125123289	33	16	226690971
25	13	165470930	34	14	199026621
26	13	169068613	35	14	205014664
27	16	214393636	36	12	175733360
28	16	213891504	*	*	*

The total number of centers associated with a = 8 is 214. The total number of the squares is
27 807 342 954.

[7] As summary, we have 2738 centers, which are neither symmetric nor semi symmetric and have negative determinant. Based on the data in [6], [7] and in this paper we state: the number of the squares associated with these centers is represented in the following list:
List of the number with a=1,…,16

a	Center	number	a	Center	number
1	255	3852302267	9	183	2412601781
2	270	4066291938	10	152	1762923729
3	279	3962819394	11	121	1524419994
4	282	4086636847	12	87	1106188872
5	280	3966691562	13	55	7037768734
6	266	3715782866	14	33	4328085633
7	242	3313390172	15	15	2059934349
8	214	2780734295	16	4	618706214

When we sum all numbers together we conclude that: the number of the squares with negative center is
379 552 332 175.
Hence, the total number of the squares with negative center is
379552332175 *16 = 6 072 837 314
There are 232 centers of the natural positive determinant four corner magic squares. According to [9] there are
30 350 772 825 * 16 = 485 612 365 200
squares of this type. There are 153 possible symmetric centers of the natural four corner magic squares. According to [11] there are
28 634 584 244 * 16 = 458 153 347 904
different natural four corner magic squares with symmetric center. There are 306 possible semi symmetric centers of the natural four corner magic squares. According to [10] there are
101 425 060 998 * 16 = 1 622 800 975 968
different natural four corner magic squares with semi symmetric center. Hence, there are
8 639 404 003 872
different four corner magic squares of order six.
The problem of counting the number of squares of order six has been now completely solved. We see that the maximum number of squares for a fixed center is the number generated by the semi symmetric center
a = 17, b = 20, e = 18
It is 398369256. Further, the minimum number of squares for a fixed center is the number generated by the symmetric center
a = 1, b = 35, e = 2.
It is 80012582.

4. GENERAL BALANCED MAGIC SQUARE

4.1 The case of any order
We distinguish between even and odd order for a magic square in the definition. A $2l$ by $2l$ magic square is called balanced iff

$a_{k,k} + a_{k,2l+1-k} + a_{2l+1-k,k} + a_{2l+1-k,2l+1-k} = 2s$, for all $1 \leq k \leq l$, where $s = (2l)^2 + 1$. This means the four corners of the center squares sum up to $2s$ each.

In case of $2l+1$ by $2l+1$ magic square then we require that

$a_{k,k} + a_{k,2(l+1)-k} + a_{2(l+1)-k,k} + a_{2(l+1)-k,\,2(l+1)-k} = 4s$, for all $1 \le k \le l$,

$a_{l+1,l+1} = s, \quad s = 2l^2+2l+1.$

It is well-known that the following structure

h	i	j	2s – h – i – j
v	q	2s – v – q – l	l
6s –(l+g+i+j)–2(q+h+v)	2s – g – q	l+v+g+q-2s	2(h+q)+i+j+v+g – 4s
l+v+g+h+i+j+2q – 4s	g – i	2s – g – j	4s – h – 2q– l – v – g

is a general structure of the magic square 4 by 4. Here, the magic constant is 2s. So, we can say that all magic square 4 by 4 are balanced. Also, all natural magic square 3 by 3 are balanced, while there are according to our calculations just

830 396 · 4 · 8 = 26 572 672

balanced natural magic square 5 by 5. The number of balanced natural magic square of orders 6 and higher is still open.

Actually, it is well-known that the following structure

A	B	D	E	F	f	G
H	J	i	j	K	l	L
m	n	o	p	q	r	R
t	u	W	s	2s–W	2s – u	2s–t
2s – R	2s – r	2s – q	2s – p	2s – o	2s – n	2s – m
2s – L	2s – l	2s – K	2s – j	2s – i	2s – J	2s – H
2s – G	2s – f	2s – F	2s – E	2s – D	2s – B	2s – A

where

A = 2i + 6s + l – 2f – 2r – 4m + t – o – p – u + 2j, B=n – f + q + r + o + j – 3s,
D = u – j + s + m + n – i – l, E = 2f + 6s – 2i + 2m – t – 2j – 2n – q – o,
F = 3s – l – i + r + m – t – j, G = 2i + l + t + j + o + p – 6s,
H=f + r + 2m + o + p + u – j – i – 2s – l – t, J = 4s + 2f + l – q – o – j – u – 2n,
K=3s + i – 2f – r – 2m + t – p + j + n, L= 2s + n – l – f – i + q,
R=7s – m – n – o – p – q – r, W = 6s + q – o + j – 2f – 2m – p – u.

is a general structure of the symmetric pandiagonal magic square 7 by 7. Here, the magic constant is 7s. Of course, such magic squares are balanced. The number of natural symmetric pandiagonal magic square is

20 190 684.

We call a matrix (a_{ij}) a balanced magic 6x6 square if

$a_{11} + a_{61} + a_{16} + a_{66} = 2s$

$a_{22} + a_{25} + a_{52} + a_{55} = 2s$

$a_{33} + a_{43} + a_{34} + a_{44} = 2s$

Compared with a general magic square we have 3 additional equations (the corner sum of the center 2x2, 4x4, 6x6 square), but only 2 are linearly independent in the whole set of equations. Hence, there are 21 independent variables for balanced magic 6x6 squares.

4.2 The case of order six

We present a general form of balanced magic 6x6 squares as follows:

a	j	C	D	f	3s – a – j – c – D – f
h	n	K	l	m	3s – K – l – m – n – h
b	r	u	v	t	3s – u – v – x – r – b
q	p	z	2s – u – v – z	y	s – q – p + u + v – y
g	o	i	x	2s – n – o – m	m + n + s – i – x – g
T	Y	R	E	G	m + v + z + o – a – s

where

D = 3s – 2a – h – b – q – g – j – c – f + m + v + z + o,
E = s – D – l – x + u + z, G = s – f – t – y + n + o,
K = 4s – (l + p + r + t + x + y + i), T = 3s – a – h – b – q – g,
Y = 3s – j – n – o – p – r, R = 3s – c – K– i – u – z.

We notice that it has 21 free variables.

For a semi-pandiagonal magic square we additionally require the following sums for 2 broken diagonals:

$a_{31} + a_{22} + a_{13} + a_{64} + a_{55} + a_{46} = 3s$

$a_{41} + a_{52} + a_{63} + a_{14} + a_{25} + a_{36} = 3s$

If one of these equations is satisfied in a balanced magic square of order 6 then the other equation is satisfied, too. Therefore we get only one additional free variable. From

b + n + c + E + (2s – n – o – m) + (s – q – p + u + v – y) = 3s (3)

we obtain

p = 2(a + b + c + u – m – o – s) + f + g + h + j – l – x – y (4)

By using this value for p we obtain a general form for balanced semi-pandiagonal magic squares of order 6. It has 20 free variables, which coincide with the previous definition.

4.3 The case of any order seven

How many natural balanced pandiagonal magic squares 7 by 7 are there? In order to solve this open problem we consider first the general form of such squares:

A	B	c	d	e	f	η
G	h	J	K	l	m	σ
n	o	p	q	R	i	ξ
u	v	ε	s	x	z	4s−t−u−d
Γ	τ	Σ	4s−q−x− ε	4s−p−R − Σ	y	ϑ
Θ	δ	ϕ	4s−K−v−z	ψ	4s−m−h − δ	α
λ	ρ	Π	t	w	μ	4s−A−λ−η

$A = f-d-j+m-o-p-u+y+z+\mu-\phi+2s$, $B = u-f-h-n-q-c-y+\phi+5s$

$G = c+e+f+x+y+z-d-j+m-2o-p-u$, $K = o-h-m-c+i-t-e+4s$

$R = c-d+h-j-l+m-2o-p-q-2i+t+e+5s$, $w = i-d-t-u+y+z-e+2s$,

$\Gamma = d-f+j-n-i+t+u-v-y-z-\mu+\phi+3s$,

$\Theta = j-f-c-m+2o+p+i-t+v-e$, $\psi = d+u-c-h-n-y+3s$,

$\rho = d+j+l+q-v-z-\mu$, $\tau = c+h+n-d-j-l-o-i-u-\phi+5s$,

$\varepsilon = d+t-v-x-z+2s$, $\Sigma = l+x+z-h-c-n-p+i-t+2s$,

$\delta = f+i+y+z+\mu-h-3s$, $\eta = h+j+n+o+p+q-f-m-z-\mu-e$,

$\lambda = d-m+o+p-x-y-z+2s$, $\Pi = h+n+v-d-j-l-i-\phi+3s$,

$\vartheta = f+e+m+t+\mu-j-o-2s$, $\sigma = 7s-G-h-j-K-l-m$,

$\alpha = c+f+h+m+n+y+z-d-j-o-p-u-\phi$, $\xi = 7s-n-o-p-q-R-i$.

It has 21 independent variables. This seems to be a more difficult problem than counting the balanced squares 6 by 6. We shall note that we have now a wider range of numbers, namely from 1 to 49, which requires also more calculations.

We consider a special case of this structure. It has a special 5 by 5 square in the center. The general form of it is

c	a	h	g	p	q	50−L
v	W	N	I	u	50−x	50−v
e	y	k	D	75−k−D	50− y	50−e
b	z	100−(2k+D)	25	2k+D−50	50−z	50−b
t	F	k+D−25	50−D	50−k	50−F	50−t
m	x	50−u	50−I	50−N	50−W	50−m
L	s	j	50−g	O	R	50−c

$a = 2L+v+b-g+j+m-p-q+t-2u+x-I-W+e-25$,

$c = 175-v-b-m-t-e-L$,

$F = g-v-b-y-2L-j-m+p+q-s-t+2u-2x-z+I-e+200$,

$h = 2u-j-x+I+W-25$, $N = x-u-I-W+75$,

$O = x-2u-p-I-W+125$, $R = g-v-b-2L-j-m+p-s-t+2u-x+I+W-e+125$.

It has 18 independent variables. We see that the center is a 3 by 3 magic square. The center is independent of the other cells. We fix the center and then count the squares such that $W < x < 50-x$, $W < 50-W$.

We restate this condition as $W < x < 25$.

5.CONCLUSION

5.1. Conclusion

We have considered many types of magic squares. We talked about the way how to calculate their number. Sometimes we had to consider subsets of the general class in order to be able to calculate the number in reasonable time.

5.2. Suggestion

We may try to find more property preserving transformations. This will help by reducing the run time. We shall also try to write the codes for the computations by using nested loops in such way, which allows parallel computing. By doing this we can reduce the time of calculations. Also, the way of checking the conditions of the square can be written in a the manner of backtracking, which reduces the time of calculations.

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