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Application of the Homotopy Perturbation method to solve nonlinear oscillatory differential equations

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ABSTRACT

This research article explores on the validation of homotopy perturbation procedure to solve the ordinary differential equations i) nonlinear electrical phenomenon in nonlinear inductor circuit and ii) movement of a ball bearing oscillating in bent tube with smoothness where restoring force is directly proportional to the displacement's cube. This approach gives periodic solution and period of the motion as a function of amplitude of oscillation also. The Homotopy method works very well for the cubic oscillator and good agreement of the approximate frequencies with the exact ones. The significant features of homotopy method are simplicity and its excellent accuracy for the complete range of oscillation amplitude values. This technique is very effective and convenient for solving truly nonlinear oscillatory systems. Here a comparative analysis has been made between two methods namely homotopy perturbation and the formal methods. More interestingly an isomorphism can be seen between the two methods of solution in this paper. Furthermore the uniformity of the validation of solution technique by homotopy perturbation process on entire domain has been depicted and it has been observed that the methods proposed here can be applied to strong systems which are not linear as well as for the weak systems which are considerably weak.

Key words: Homotopy perturbation method (HPM); Modified differential transform method (MDTM), Phase diagram, Truly nonlinear system, Perturbation Method (PM)

1. INTRODUCTION

The study of nonlinear systems has a large number of applications in many branches of applicable mathematics namely physical sciences, management and engineering sciences. The nonlinear. The nonlinear problems are too complex to get the analytic solution but their numerical solution can be obtained by some techniques. On the other side finding the governing differential equations and solving by using the different techniques is also interesting one in mechanics and mathematics. Most of the researchers are working on finding the solutions to these nonlinear equations by using the methods such as Variational iteration method, Modified differential transform method and Homotopy analysis method and so on.

Several researchers have been trying to solve nonlinear systems possessing low nonlinearity. As the small parameter takes a vital role in the PM. This parameter decides the accuracy and the validation of the PM.

2. HOMOTOPY PERTURBATION METHOD

He [10] proposed a technique namely hompotopy perturbation which is a amalgamation of traditional PM and homotopy. In employing this technique, the required solution is treated as a sum of an infinite series. Besides this infinite series converges to the exact solution rapidly. A large number of different LDE and NonLDE can be solved by using HPM. To illustrate the Homotopy method let us choose a NonLDE

$$E(x)+g(s)=0,\ s\in\Omega \qquad (1)$$

With boundary conditions

$$F(x, \partial x/\partial n)=0\ ,\ s\in\Gamma \qquad (2)$$

Symbol	Description
E	GENERAL DIFFERENTIAL OPERATOR
F	BOUNDARY OPERATOR
x	ANALYTIC MAPPING
s	DOMAIN'S BOUNDARY

E is divided into P and Q where Q and P are nonlinear and linear respectively. Therefore (1) is put as

$$P(x)+Q(x)-g(s)=0 \qquad (3)$$

Liao (1) presented his homotopy technique as given below .He, built a homotopy as

$y(s,q)$ maps $\Omega[0,1]$ to $\Re$ following,

$$H(y,q)=P(y)-P(z_0)+qP(z_0)+q[Q(y)-g(s)]=0 \qquad (4)$$

q is embedding parameter,

q lies between 0 and 1,

z_0 is the starting approximation of (1)

Boundary conditions are satisfied by q and z_0

$$(4)\Rightarrow H(y,0)=P(y)-P(z_0) \qquad (5)$$

$$H(y,1)=E(y)-g(s)=0 \qquad (6)$$

The two process of variations namely "q moves from 0 to 1" and "q moves from z_0 to $z(s)$" are one and the same and this phenomenon, in TOPLOGY, is depicted by DEFORMATION.

Besides $E(y)-g(s)$ is called homotopic. The embedding parameter is introduced in such a way that it is not effected by any feigned factor and it is treated for minute parameters $q\in[0,1]$. Then solution of the equation (4) is written in specific form as

$$y=y+qy_1+q^2y_2+..... \qquad (7)$$

The nearest solution of equation (1) is given by

$$z=\lim_{q\to 1} y=\sum_{l=0}^{\infty} y_l \qquad (8)$$

3. ILLUSTRATIVE EXAMPLES

Example-1

A simple example of electrical circuit is a charged capacitor C connected to a coil of N turns wrapped around an iron core. The current (i) versus flux (ϕ) relation for the iron core inductor has the form , In the following Figure 1 there is a electrical circuit which possesses a charged capacitor C and is connected to a coil of N. ϕ and i maintains the following relationship.

$$P_0^{-1}\phi N+\phi^3E=i \qquad (9)$$

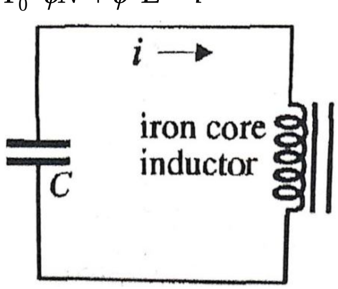

Figure 1 . Nonlinear inductor circuit

If the cube term does not appear then the linear relationship $Q\phi=P_0^i$

symbol	Description
P	Coil's own inductance
ϕ	flux

The appearance of iron core generates the cube term. The compact form of the nonlinearity is completely decided by the core's nature.

From Kirchhoff's law one can see

$$qC^{-1}+N\phi'=0 \qquad (10)$$

As $i=q'$, by differentiating (10) and considering (9) one can see

$$\ddot{\phi}+\alpha\phi+\beta\phi^3=0 \qquad (11)$$

Where $\alpha=\dfrac{1}{L_0C}$; $\beta=\dfrac{A}{NC}$;

In the above equation (11) if $\alpha=1;$ and $\beta=0.3$ then the equation (11) becomes

$$\ddot{\phi}+\phi+0.3\phi^3=0$$

And by taking the initial conditions as

$$\phi(0)=A \text{ and } \left(\frac{d\phi}{dt}\right)_{t=0}=0$$

Applying the homotopy perturbation method to the equation (11) with the following boundary conditions

$$\phi(0)=A \text{ and } \left(\frac{d\phi}{dt}\right)_{t=0}=0 \qquad (12)$$

Now a homotopy $\Omega[0,1]\to R$ is bulid with the following equations

$$P(y)-P(\phi_0)+qL(\phi_0)+qv^3=0 \qquad (13)$$

Where $P(\phi)=\ddot{\phi}+\phi$ and $Q(\phi)=0.3\phi^3$

Assuming the initial approximation of equation (1) is of the form

$$\phi_0(t)=A\cos(\alpha t) \qquad (14)$$

here $\alpha(\varepsilon)\neq 0$ not known constant and α at 0 is unity. The nearest solution of (3) can put as

$$y=y_0+qv_1+q^2v_2+... \qquad (15)$$

And $$\phi=\lim_{q\to 1} y=\sum_{l=0}^{\infty} y_l \qquad (16)$$

Substituting (15) in (13) & comparing like powers of q

$$P(y_0)-P(\phi_0)=0\ ,\ \ y_0(0)=A, y_0^{'}(0)=0 \qquad (17)$$

$$P(y_1)+P(\phi_0)+0.3y_0^{\ 3}=0,\ \ y_1(0)=y_1^{'}(0)=0 \qquad (18)$$

From equation (4) we have $y_0=\phi_0=A\cos\alpha t$

Then from the equation (18) we have

$$\frac{d^2 y_1}{dt^2} + y_1 + (-\alpha^2 + 1 + 0.9\frac{A^2}{4})A\cos\alpha t + \frac{0.3A^3}{4}\cos 3\alpha t = 0 \quad (19)$$

Solving the equation (19) one gets

$$v_1(t) = (-\alpha^2 + 1 + \frac{0.9A^3}{4})\frac{A}{(\alpha^2 - 1)}\cos\alpha t + \frac{0.3A^3}{4(9\alpha^2 - 1)}\cos 3\alpha t \quad (20)$$

To remove the secular term this can occur in the coming computations, we put α equals 0. Then the first order approximation for the equation (19) is

$\phi(t) = y_0(t) + y_1(t)$ Then we have

$$\phi(t) = \frac{0.9A^3}{4(\alpha^2 - 1)}\cos\alpha t + \frac{0.3A^3}{4(9\alpha^2 - 1)}\cos 3\alpha t \quad (21)$$

In (15) when A=1, then the solution becomes

$$\phi(t) = 0.99255\cos(1.10756)t + 0.01117192\cos(2.72025)t \quad (22)$$

3.1 Generation of phase diagram

The restoring force function in the equation of motion (1) considering $\alpha = 1$ and $\gamma = 0.3$ is

$$f(\phi) = \phi + 0.3\phi^3, \quad (23)$$

Which is a cubic polynomial. For both +ve and –ve amplitudes. The nature of oscillations is one and the same. DE (1)'s singular point is the phase diagram (i.e ϕ' versus ϕ curve) obtained from the roots of $g(y)$ is (0,0). The derivative of he behavior of $g(\phi)$ wrt ϕ is $g'(\phi) = 1 + 0.3\phi^2$ and $g'(0) = 1 + 0.3\phi^2$ and $g'(0) = 1 > 0$ which implies that the singular point (0,0) becomes a centre. If $g'(\phi^*) < 0$ then ϕ^* is a saddle point. In the present problem there is no saddle point and hence no separatrix formation. This implies that there is no boundary formation between the stable ad unstable regions of the motion of the oscillator.

Define the potential energy function,

$$I(\phi) = \int_0^e f(\xi)\,d\xi = \frac{\phi^2}{2} + \frac{0.3\phi^4}{4},$$

(which implies that $\frac{dI}{d\phi} = f(\phi)$), the equation of motion (1) can be written in the form

$$\frac{d^2\phi}{dt^2} + \frac{dI}{d\phi} = 0 \quad (24)$$

Multiplying equation (24) by $2\dot{\phi}$ and applying the initial conditions (2), one gets after integration

$$(\dot{\phi})^2 - 1 + 2\{I(\phi) - I(0)\} = 0 \quad (25)$$

Equation (25) for the DUFFING equation of motion (1) with the starting conditions (2) are written by

$$(\dot{\phi})^2 = 1 - \phi^2 - 0.15\phi^4$$
$$= (1 + 0.13246\phi^2)(1 - 1.13246\phi^2) \quad (26)$$

Equation (26) represents the phase diagram (i.e., $\dot{\phi}$ versus ϕ) for the differential equation (1) with initial conditions (2). $\dot{\phi}$ Versus ϕ plot generated from equation (26) shows closed boundary, which implies the existence of the periodic solution. Equation (26) gives equal magnitude of the positive and negative amplitudes (i.e. $\phi = \pm 0.9397, \dot{\phi} = 0$).

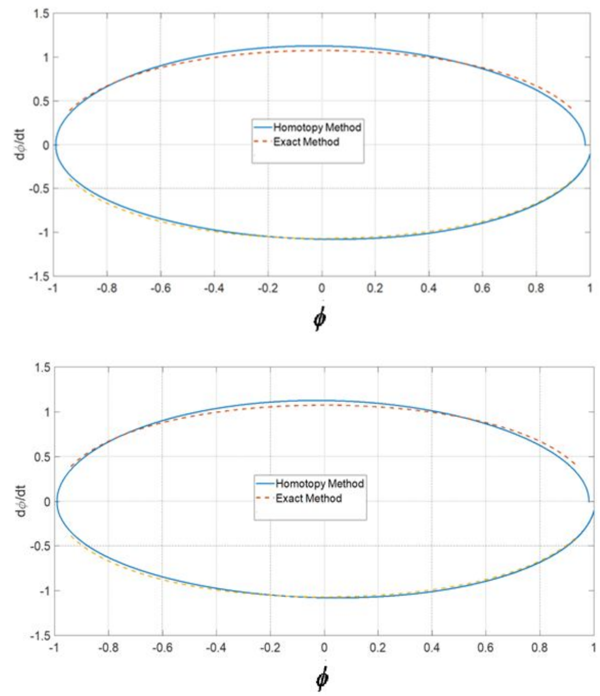

Figure 2. Comparison of the Phase diagram for the solutions obtained by the Homotopy method to the exact method

Example 2:

Suppose there is a curved glass tube and a ball bearing oscillates in it. We know that the restoring force completely depends on the displacement's cube. We neglect frictional losses here. The controlling expressions are

$$\frac{dz}{dt^2} + cz^3 = 0$$

Where c=1

At t=0 and distance u_0 the ball bearing falls from rest and the auxiliary parameters

$$z(0) = A, \ \frac{dz}{dt} \ is \ zero \ at \ t = 0$$

Whineray built a mechanical oscillator and in that the restoring force is almost close to the proposition of displacement's cube. He constructed, with springs, an air track oscillator which is linear. Here the springs are put at right angles to route rather than on it. This can demonstrate effectively the principles of the cube law oscillators. By applying the homotopy method to the above equation it is modified as

$$\frac{d^2z}{dt^2} + z = z - z^3 \text{ with}$$

$$z(0) = A \ and \ \frac{dz}{dt} = 0 \ at \ t = 0$$

We consider $\Omega[0,1] \to R$ satisfying

$$P(y) - P(z_0) + qP(z_0) + qy^3 = 0 \qquad (27)$$

Where $P(z) = \frac{d^2z}{dt^2} + z$ and $Q(z) = 0.3z^3$ Assuming the initial approximation of equation (1) as

$$z_0(t) = A\cos(\alpha t) \qquad (28)$$

Where $\alpha(\delta) \neq 0$, unknown, constant, $\alpha(\delta) = 0.$

The nearest solution of equation (3) is put as

$$y = y_0 + qy_1 + q^2 y_2 + ... \qquad (29)$$

$$z = \lim_{q \to 1} y = y_0 + y_1 + y_2 + ... \qquad (30)$$

Put (15) into (13) and compare the expressions with equal powers of q, one can see

$$P(y_0) - P(z_0) = 0, \ y_0(0) = A, y_0'(0) = 0 \qquad (31)$$

$$P(y_1) + P(z_0) = y_0 - {y_0}^3, \qquad (32)$$

$$y_1(0) \quad and \quad y_1'(0) \quad are \quad zeros$$

From (4) one can see $y_0 = z_0 = A\cos\alpha t$,then from the equation (18) we have

$$\frac{d^2 y_1}{dt^2} + y_1 + -A\alpha^2 \cos\alpha t = -\frac{A^3}{4}\cos 3\alpha t - \frac{3A^3}{4}\cos\alpha t \qquad (33)$$

Solving (32) one can see

$$y_1(t) = \left(\alpha^2 - \frac{3A^2}{4}\right)\left(\frac{A}{(-\alpha^2+1)}\cos\alpha t - \frac{A^3}{4(-9\alpha^2-1)}\cos 3\alpha t\right) \qquad (34)$$

Secular expression might come in the foregoing iterations. To remove this makes the cost's coefficient as

$$\alpha = \sqrt{\frac{3A^2}{4}}$$

Then the first order approximation for the equation (27) is

$$z(t) = y_0(t) + y_1(t)$$

Then we have

$$z(t) = A\cos\alpha t + \frac{A^3}{4}\frac{1}{(9\alpha^2+1)}\cos 3\alpha t$$

If we put A=1 then

$$z(t) = \cos 0.8660t + 0.0322\cos(2.598)t \qquad (35)$$

3.2 Generation of PHASE DIAGRAM

To study the fit of the SOLUTION (35) of the truly NLODE, the phase diagrams are to be constructed. Figure1 depicts the differences between PHASE DIAGRAMS come out of equation (7), (28).The true value of dz/dt are plus or minus 0.7071 at the instant u is zero. But from the modified differential transform method y' values are computed as -0.850 and 0.9393 respectively. There are known to be distinct in their magnitudes for the particular greatest +ve amplitude of units the real PHASE DIAGRAM depicts the –ve amplitude as minus1.

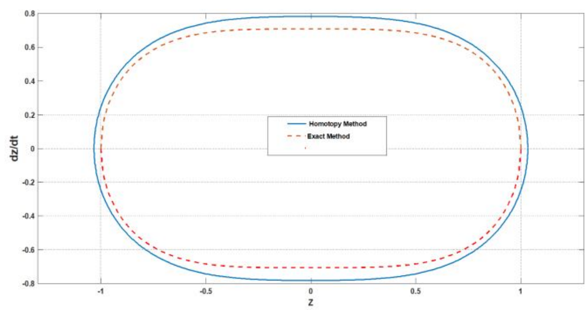

Figure 3. Comparison of the Phase diagram for the solutions got by HPM and the formal method

4. CONCLUSION

The solutions got by HPM for above two applications, have coincidence with the solutions obtained by the direct method. For the first example the solution obtained by the homotopy method is close to the solution obtained by the direct method whereas for the second example the solution is near to the exact solution. Hence the homotopy method is able to provide the periodic solutions of the nonlinear differential equations. Appendixes, if needed, appear before the acknowledgment.

REFERENCES

1. I.Khatami, E.Zahedi, M.Zahedi, **Efficient Solution of Nonlinear Duffing Oscillator**, *Journal of Applied and Computational Mechanics*, vol.6 (2):219-234, 2020.
2. J.Peter Praveen, B.Nageswara Rao, **On the periodic solutions of the Helmholtz equations using the modified differential transform method**, *MAYFEB Journal of Mathematics*, Vol.2:7-16, 2016.
3. J.Peter Praveen, B.Nageswara Rao, **Applicability of the**

MDTM to obtain the periodic Solution of the Duffing type equations of motion, *Asian Journal mathematics and Computer Research*, Vol. 18(1):19-27, 2017.

4. J.Peter Praveen, B.Nageswara Rao, **Uncertainties in the Periodic Solution of a Truly Non-linear Oscillator Differential Equation Using MDTM**, *International Journal of Applied and Computational Mathematics*, Vol.5 (4) doi.org/10.1007/s40819-019-0702-3
5. M.Turkyilmazoglu, The **selection of paper auxiliary parameters in the Homotopy analysis method. A case study: Uniform solution of undamped Duffing equation**, *Applied Mathematics*, Vol.2:783-790, 2017.
6. Z.Azimzadeh, A R Vahidi and E Babolian **Exact solutions for nonlinear Duffing equations by He's Homotopy Perturbation method**, *Indian Journal of Physics* Vol.86 (8):721-726, 2012
7. Y.Khan Y, M Akbarzade and A Kargar ,Coupling of **Homotopy and the variational approach for a Conservative oscillator with strong odd-nonlinearity**, *Scientia Iranica* vol.19 (3):417-422 , 2012
8. M M Zhang, Ben-Liwang, Xian-RenKong and A-Yang Xiao, **Application of Homotopy perturbation method for harmonically forced Duffing systems,** *Applied Mechanics and Materials,* Vol.110-116:.2277-2283 ,2012
9. S.Liao, Homotopy **Analysis Method in Nonlinear Differential Equations**, Springer, Berlin, Heidelberg, https://doi.org/10.1007/978-3-642-25132-0
10. S R Saratha, Vijeta Iyer and R Manju R , **Subspace Homotopy Methods for Solving Nonlinear Equations**, *International Journal Recent Technology and Engineering*, Vol. 7(4S):1-4 , 2018
11. Huang S,Rafiq A , Md. Rizwan Shahzad and Faisal Ali , **New higher order iterative methods for solving Nonlinear equations**, *Hacettepe J Mathematics and Statistics*, Vol.47 (1): 77-91, 2018.
12. Wenrui Hao, **A Homotopy Method for Parameter Estimation of Nonlinear Differential Equations with Multiple, Optima**, *Journal Scientific Computing* 74:1314-1324,2018
13. E Babolian, A R Vahidi, Z Azimzabelu Z, An **Improvement to the Homotopy perturbation method for solving nonlinear Duffing equations**, Bulletin of the Malaysian Mathematical Sciences Society,Vol.41(2):1105-1117 ,2018.
14. Ihtisham ul Haq , **Analytical Approximate Solution of Non-Linear Problem by Homotopy Perturbation Method (HPM)**, *Matrix Science Mathematic* , Vol.1:20-24
15. J. Peter Praveen, R. Donthi, S.V.Prasad, B. Mahaboob, B.Venkateswarlu, **A glance on the estimation of Cobb-Douglas production functional model,** *AIP Conference Proceedings*, 2177 (1), 020067. 2019
16. J.Peter Praveen, B. Mahaboob, R.Donthi, S.V. Prasad, B.Venkateswarlu, **On stochastic linear regression model Selection,** *AIP Conference Proceedings*, 2177 (1), 020068, 2019.
17. B.Mahaboob, J.Peter Praveen, R.Donthi, S.V.Prasad, B.Venkateswarlu B., **Criteria for selection of stochastic linear model selection**, *AIP Conference Proceedings*, 2177 , 020041,2019.
18. R.Donthi, S.V.Prasad, B.Mahaboob, J.Peter Praveen, B.Venkateswarlu. **Estimation methods of nonlinear regression models**, *AIP Conference Proceedings*, 2177 (1), 020081, 2019.
19. R.Donthi, J.Peter Praveen, S.V.Prasad, B.Mahaboob, B.Venkateswarlu, **Numerical techniques of nonlinear regression model estimation**, AIP Conference *Proceedings*, 2177 (1), 020082. 2019.
20. B.Mahaboob, S.V.Prasad, J.Peter Praveen, R.Donthi, B.Venkateswarlu, On **misspecification tests for Stochastic linear regression model**, *AIP Conference Proceedings*, 2177 (1), 020039. 2019.
21. B.Mahaboob, K.A.Ajmath, B.Venkateswarlu, C.Narayana, J.Peter Praveen., On **Cobb-Douglas Production function model,** *AIP Conference Proceedings*, 2177 (1), 020040. 2019.
22. B.Mahaboob, K.A.Azmath, B.Venkateswarlu, C.Narayana, B.M.Rao, An **evaluation in generalized LSE of linearized stochastic statistical model with Non-spherical errors**, *AIP Conference Proceedings*, 2177(1), 020038. 2019.
23. T.S.Rao., G.S. Kumar, Ch.Vasavi, B.V.A. Rao., **On the Controllability of fuzzy difference control systems**, *International Journal of Civil Engineering and Technology*, 8(12), PP.723-732,2017.
24. Ch.Vasavi, G.S.Kumar, T.S.Rao, B.V.A.Rao., **Application of fuzzy differential equations for cooling Problems**, *International Journal of Mechanical Engineering and Technology*, 8(12), PP.712-721,2017.
25. J. Peter Praveen, B. Nageswara Rao, B.Mahaboob, M.Rajaiah, Y.Harnath, C.Narayana, **On the Simulation of Beck Column through a Simple Xiong-Wang-Tabarrok Experimental Model of Centripetally Loaded Column,** *International Journal of Emerging Trends in Engineering Research*,vol.8(9),pp.5100-5103,2020
26. J.Peter Praveen, B.Nageswara Rao, B.Mahaboob, B.V.Appa Rao, **An Application of Laplace Transform, Inverse Laplace Transform and Pade's Approximant in the Periodic Solution of Duffing Equation of**

Motion, *International Journal of Emerging Trends in Engineering Research,* vol.8 (9), pp.5979-5983, 2020

27. J. Peter Praveen, B. Nageswara Rao, Y. Harnath, B.V. Appa Rao, C. Narayana, and B. Mahaboob, "**Existence of solution of the nonlinear differential equation in the modelling of eardrum by using homotopy perturbation method**", *Advances in Mathematics: Scientific Journal*, Vol.9, No.7, pp.4853-4862 ,2020.
28. J. Peter Praveen, B. Nageswara Rao, B. Mahaboob, C.Rajaiah,Y. Harnath, , C. Narayana, **Series Decomposition method for asymmetric nonlinear Oscillations**, *Advances in Mathematics: Scientific Journal*, Vol.9, No.10, pp.8069-8076, 2020.