ISSN 2278-3091
Volume 9 No.2, March -April 2020
International Journal of Advanced Trends in Computer Science and Engineering
Available Online at http://www.warse.org/IJATCSE/static/pdf/file/ijatcse87922020.pdf
https://doi.org/10.30534/ijatcse/2020/87922020

Solutions for Nonlinear System of Fractional Integro–differential Equations with Non-separated Integral Coupled Boundary Conditions

Raad N. Butris [1], Ava Sh. Rafeeq [2]
[1]Department of Mathematics, College of Basic Education,Duhok University, Duhok, Kurdistan region-Iraq, raad.butris@uod.ac
[2]Department of Mathematics, College of science,Duhok University, Duhok, Kurdistan region-Iraq, ava.rafeeq@uoz.edu.krd

ABSTRACT

In this paper, we investigate the existence and uniqueness of solutions for a system of fractional integro-differential equations with non-separated integral coupled boundary conditions. Our results are based on the nonlinear alternative of Leray-Schauder type to study the existence of at least one continuous solution to fractional integro- differential system with non-separated integral coupled boundary conditions and uniqueness continuous solution using the Banach's fixed-point theorem.The main results are well illustrated with the aid of an example.

Key words: Caputo fractional derivative, fractional integro-differential , integral coupled boundary conditions, fixed-point theorem ,Leray-Schauder Alternate.

1. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, economics, control theory, signal and image processing, blood flow phenomena, *etc.* [9,10,13].The theory and applications of the fractional differential equations have recently been addressed by several researchers for a variety of problems, we refer the reader to [1,4,8] and the references cited therein. Momani and Hadid have investigated the local and global existence theorem of both fractional differential equation and fractional integro-differential equations; see [7,11].

The study of a coupled system of fractional order is also very significant because this kind of system can often occur in applications. The reader is referred to the papers [2,5,14,15], and the references cited therein.

In [3], it is studied the following nonlinear problem involving nonlinear integral conditions:

$$^{c}D^{\alpha}y(t) = f(t, y(t)), \quad t \in [0,T], \qquad 1 < \alpha \le 2$$
$$y(0) + y'(0) = \int_0^T g(s, y(s))ds,$$
$$y(T) + y'(T) = \int_0^T h(s, y(s))ds,$$

Here, f , g and $h : [0,T] \times E \to E$ are given functions that satisfy suitable assumptions and E is a Banach space. By means of the technique associated with measures of non-compactness and the fixed-point theorem of Monch type, it is proved the existence of solutions of the problem.

In [12], investigated a boundary value problem of first order fractional differential equations with Riemann-Liouville integral boundary conditions of different order given by

$$^{c}D^{\alpha}_{0+}u(t) = f(t, u(t), v(t)), \quad t \in [0,1],$$
$$^{c}D^{\beta}_{0+}v(t) = g(t, u(t), v(t)), \quad t \in [0,1],$$
$$u(0) = \gamma I^{p}u(\delta) = \gamma \int_0^{\delta} \frac{(\delta - s)^{p-1}}{\Gamma(p)} u(s)ds, \quad 0 < \delta < 1$$
$$v(0) = \sigma I^{q}v(\varepsilon) = \sigma \int_0^{\varepsilon} \frac{(\varepsilon - s)^{q-1}}{\Gamma(q)} v(s)ds, \quad 0 < \varepsilon < 1$$

where $^{c}D^{\alpha}_{0+}$, $^{c}D^{\beta}_{0+}$, denote the Caputo fractional derivatives, $1 < \alpha , \beta \le 2, f , g \in C([0,1] \times R^2, R^1), and\ p, q, \gamma , \sigma \in R.$

In this paper, we consider the fractional integro-differential equation

$$\left.\begin{aligned} ^{c}D^{\alpha}_{0+}x(t) &= f\left(t, \beta(\tau,\alpha), x(t), \int_0^{\omega(t)} K(t,s)(x(s) - y(s))ds\right) \\ ^{c}D^{\gamma}_{0+}y(t) &= g\left(t, \beta(\tau,\gamma), y(t), \int_0^{\varphi(t)} R(t,s)(x(s) - y(s))ds\right) \end{aligned}\right\} \quad \dots (1.1)$$

with non-separated integral coupled boundary conditions

$$\left.\begin{aligned} ax(0) + bx(T) &= \int_0^T h_1(y(s))ds, \quad with \ \ a + b \neq 0 \\ cy(0) + dy(T) &= \int_0^T h_2(x(s))ds, \quad with \ \ c + d \neq 0 \end{aligned}\right\} \quad \dots (1.2)$$

for all $t \in [0,T], a, b, c, d \in R^1$ where ${}^cD_{0+}^{\alpha}$, ${}^cD_{0+}^{\gamma}$, denote the Caputo fractional derivatives, $0 < \alpha, \gamma \leq 1$, also $\beta(\tau,\alpha)$ and $\beta(\tau,\gamma)$ are said to be special functions provided that (Beta function).

2. PRELIMINARIES

Let us introduce the space $X = \{x(t)|x(t) \in C[0,T]\}$ endowed with the norm $\|x\| = max\{|x(t)|; t \in [0,T]\}$ obviously, $(x, \|.\|)$ is a Banach space. Also let $Y = \{y(t)|y(t) \in C[0,T]\}$ endowed with the norm $\|y\| = max\{|y(t)|; t \in [0,T]\}$. The product space $((X \times Y), \|(x,y)\|$ is also Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$.
Now, we call some basic useful definitions and fundamental facts of fractional calculus

Definition 2.1 [9] For a function f given on the interval [a, b], the Caputo fractional order derivative of f is defined by

$$ {}_0^tD^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds \quad \dots (2.1)$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, and $\Gamma(.)$ denotes the Gamma function,

i.e., $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$.

Definition 2.2[9] Let f be a function which is defined almost everywhere (a.e) on [a, b], for $\alpha > 0$, we define

$$ {}_a^bD^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} f(t)dt \quad \dots (2.2)$$

provided that the integral (Lebesgue) exists.

Lemma 2.3[9] Let $\alpha > 0$. Then the differential equation ${}_0^tD^{\alpha} f(t) = 0$ has solution

$$f(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}$$

$, c_i \in R, i = 0,1,2,..n-1,$ ... (2.3)

and

$$I^{\alpha}D^{\alpha}f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1} \quad \dots (2.4)$$

for some $c_i \in R, i = 0,1,2,..n-1$, n=[α]+1.

Lemma 2.4 [6](Leray-Schauder alternative) Let $F : E \to E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let

$$\xi(F) = \{x \in E : x = \lambda F(x) for\ some\ 0 < \lambda < 1\} \quad \dots (2.5)$$

Then either the set $\xi(F)$ is unbounded, or F has at least one fixed point.

Theorem 2.5[6]((Banach fixed point theorem*)*
Let E be a Banach space . If T is a contraction mapping on E, then T has one and only one fixed point in E.

Suppose that the functions $f, g \in C([0,T] \times \Omega \times D_1, D_2, R)$, $\Omega = (0,T] \times (0,1]$, $D_1\ and\ D_2$ are compact subset of , also ω , φ , h_1 and h_2 are continuous functions on $[0,T]$ and satisfy the following hypotheses:

H_1: There exist positive constants $k_0, k_1, k_2, l_0, l_1, l_2, M_{\beta\alpha}\ and\ M_{\beta\gamma}$ such that

$$\|f(t,\beta(\tau,\alpha),x_1,u_1) - f(t,\beta(\tau,\alpha),x_2,u_2)\| \leq M_{\beta\alpha}(k_1\|x_1 - x_2\| + k_2\|u_1 - u_2\|) \quad \dots (2.6)$$

$$\|g(t,\beta(\tau,\gamma),y_1,v_1) - g(t,\beta(\tau,\alpha),y_2,v_2)\| \leq M_{\beta\gamma}(k_1\|y_1 - y_2\| + k_2\|v_1 - v_2\|) \quad \dots (2.7)$$

where $M_{\beta\alpha} = \max_{\tau\in(0,T]} \frac{1}{\beta(\tau,\alpha)}$, $M_{\beta\gamma} = \max_{\tau\in(0,T]} \frac{1}{\beta(\tau,\gamma)}$,
$k_0 = \max_{t\in[0,T]} |f(t,\beta(\tau,\alpha),0,0)|$ and
$l_0 = \max_{t\in[0,T]} |g(t,\beta(\tau,\gamma),0,0)|$ for all $\in [0,T]$, $\tau \in (0,T]$, $x_1,x_2,y_1,y_2 \in D_1\ and\ u_1,u_2,v_1,v_2 \in D_2$.

H_2: There exist positive constants $p_0, p_1, q_0\ and\ q_1$ such that

$$\|h_1(y_1) - h_1(y_1)\| \leq p_1\|y_1 - y_2\| \quad \dots (2.8)$$

$$\|h_2(x_1) - h_2(x_2)\| \leq q_1\|x_1 - x_2\| \quad \dots (2.9)$$

$p_0 = \max_{t\in[0,T]} |h_1(0)|$ and $q_0 = \max_{t\in[0,T]} |h_2(0)|$ for all $t \in [0,T]$, $\tau \in (0,T]$, $x_1,x_2,y_1,y_2 \in D_1$.

H_3: The functions $K(t,s)$ and $R(t,s)$ satisfy the following conditions, there exist positive constants $Ks\ and\ Rs$ such that

$$\left.\begin{aligned} \int_0^{\omega(t)} \|K(t,s)\|ds \leq Ks \\ \int_0^{\gamma(t)} \|R(t,s)\|ds \leq Rs \end{aligned} \quad \text{for all } s,t \in [0,T] \right\} \dots (2.10)$$

To define the solution of the boundary value problem (1.1) and (1.2), we need the following lemma

Lemma 2.6 Let the functions $f, g \in C([0,T] \times \Omega \times D_1, D_2, R)$ be continuous functions , then the solution of the fractional integro-differential equation (1.1) with boundary condition (1.2) is the follows

$$x(t) = \frac{1}{a+b}\int_0^T h_1(y(s))ds - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, \beta(\tau,\alpha), x(s), \int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\beta(\tau,\alpha), x(s), \int_0^{\omega(s)} K(s,\mu)(x(\mu) - y(\mu))d\mu)\, ds \quad \ldots (2.11)$$

and

$$y(t) = \frac{1}{c+d}\int_0^T h_2(x(s))ds - \frac{d}{c+d}\frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} f(s, \beta(\tau,\gamma), y(s), \int_0^{\varphi(s)} R(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,\beta(\tau,\gamma), y(s), \int_0^{\varphi(s)} R(s,\mu)(x(\mu) - y(\mu))d\mu)\, ds \quad \ldots (2.12)$$

Proof: By Lemma 2.3, we reduce the problem (1.1) and (1.2) to an equivalent integral equation

$$x(t) = {}_0^tI^{\alpha} f(t) + c_0 \quad and \quad y(t) = {}_0^tI^{\gamma} g(t) + c_1 \quad \ldots (2.13)$$

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\beta(\tau,\alpha), x(s), \int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + c_0$$

and

$$y(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,\beta(\tau,\gamma), y(s), ds \int_0^{\varphi(s)} R(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + c_1$$

Applying the boundary condition (1.2), we find that

$$c_0 = \frac{1}{a+b}\int_0^T h_1(y(s))ds - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s,\beta(\tau,\alpha), x(s), \int_0^{\omega(s)} K(s,\mu)(x(\mu) - y(\mu))d\mu)\, ds$$

and

$$c_1 = \frac{1}{c+d}\int_0^T h_2(x(s))ds - \frac{d}{c+d}\frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} f(s,\beta(\tau,\gamma), y(s), \int_0^{\varphi(s)} R(s,\mu)(x(\mu) - y(\mu))d\mu)\, ds$$

Substitute c_0 and c_1 in (2.13) , we get the solutions (2.11) and (2.12).

3. MAIN RESULTS

First, we define the operator $T_{fg}: X \times Y \to X \times Y$ by

$$T_{fg}(x,y)(t) = \begin{pmatrix} T_1(x,y)(t) \\ T_2(x,y)(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{a+b}\int_0^T h_1(y(s))ds - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s,\beta(\tau,\alpha),x(s), \int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds \\ \frac{1}{c+d}\int_0^T h_2(x(s))ds - \frac{d}{c+d}\frac{1}{\Gamma(\gamma)} \int_0^T (T-s)^{\gamma-1} f(s,\beta(\tau,\gamma),y(s), \int_0^{\varphi(s)} R(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,\beta(\tau,\gamma),y(s),\int_0^{\varphi(s)} R(s,\mu)(x(\mu)-y(\mu))d\mu)\, ds \end{pmatrix} \quad \ldots (2.14)$$

Theorem 3.1 Assume that the functions $f, g \in C([0,T] \times \Omega \times D_1, D_2, R)$ be continuous functions and satisfy H_1 , H_2 , H_3 and

H_4:Let $\Omega_{xy} \subset X \times Y$ be bounded, there exist positive constants M, L, P and Q such that
$\|f(t,\beta(\tau,\alpha),x,u)\| \le M_{\beta\alpha} M$,
$\|f(t,\beta(\tau,\alpha),y,z)\| \le M_{\beta\gamma} L$
and
$\|h_1(y)\| \le P$, $\|h_2(x)\| \le Q$
where
$M_s = \min\{1 - M_1 m_2 + m_3 + M_3 M_{\beta\gamma} l_2 R_s ,\ 1 - M_3 m_4 + m_1 + M_1 M_{\beta\alpha} k_2 K_s\}$
Then the boundary value problems (1.1) and (1.2) has at least one solution.

Proof: First, we show that the operator $T_{fg}: X \times Y \to X \times Y$ is completely continuous, by continuity of the functions f , g , h_1, h_2 , ω *and* φ, then the operator T_{fg} is continuous.

Let $\Omega_{xy} \subset X \times Y$ be bounded, there exist a positive constants M and L such that
$\|f(t,\beta(\tau,\alpha),x,u)\| \le M_{\beta\alpha} M$,
$\|f(t,\beta(\tau,\alpha),y,z)\| \le M_{\beta\gamma} L$
and
$\|h_1(y)\| \le P$, $\|h_2(x)\| \le Q$
Then for any $(x,y) \in \Omega_{xy}$, we have

$$\|T_1(x,y)(t)\| \le \frac{1}{|a+b|}\int_0^T \|h_1(y(s))\|ds + \frac{b}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} \left\|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\right\| ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\right\| ds$$

So that

$$\|T_1(x,y)(t)\| \le \frac{TP}{|a+b|} + \left(\frac{|b|}{|a+b|}+1\right)\frac{T^\alpha M_{\beta\alpha} M}{\Gamma(\alpha+1)} \le \frac{TP}{|a+b|} + M_{\beta\alpha} M M_1$$

Similarly, we get

$$\|T_2(x,y)(t)\| \le \frac{TQ}{|c+d|} + \left(\frac{|d|}{|c+d|}+1\right)\frac{T^\gamma M_{\beta\gamma} L}{\Gamma(\gamma+1)} \le \frac{TQ}{|c+d|} + M_{\beta\gamma} L M_3$$

Thus, it follows from the above inequalities that that operator T_{fg} is uniformly bounded.

Next, we show that T_{fg} is equicontinuous, let $0 \le t_1 \le t_2 \le T$, then we have

$$\|T_1(x(t_2),y(t_2)) - T_1(x(t_1),y(t_1))\| \le \left\|\frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1} f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1} f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu)\,ds\right\|$$

$$\le \frac{M_{\beta\alpha}M}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right) \le \frac{M_{\beta\alpha}M}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha)$$

Also, we have

$$\|T_2(x(t_2),y(t_2)) - T_2(x(t_1),y(t_1))\| \le \frac{M_{\beta\gamma}L}{\Gamma(\gamma)}\left(\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right) \le \frac{M_{\beta\gamma}L}{\Gamma(\gamma+1)}(t_2^\alpha - t_1^\alpha)$$

Therefore the operator $T_{fg}(x,y)$ is completely continuous, as a consequence of steps together with Arzela-Ascoli theorem, we find that the operator $T_{fg}(x,y)$ is equicontinuous. Finally, it will be verified that the set

$$\xi(T_{fg}) = \{(x,y) \in X\times Y | (x,y) = \lambda T_{fg}(x,y) \quad for\ some\ 0<\lambda<1\}$$

is bounded, let $(x,y)\in\xi(T_{fg})$ then $(x,y) = \lambda T_{fg}(x,y)$, for any $t \in [0,T]$, we have $x(t) = \lambda T_1(x,y)_{(t)}$ and $y(t) = \lambda T_2(x,y)_{(t)}$, so that

$$\|x(t)\| \le \frac{T(p_1\|y(t)\|+p_0)}{|a+b|} + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\frac{|b|}{|a+b|}+1\right)\left(M_{\beta\alpha}(k_1\|x(t)\| + k_2K_s(\|x(t)\|+\|y(t)\|)) + k_0\right) \le M_1m_2\|x(t)\| + (m_1 + M_1M_{\beta\alpha}k_2K_s)\|y(t)\| + M_1k_0 + \frac{Tp_0}{|a+b|}$$

and

$$\|y(t)\| \le \frac{T(q_1\|x(t)\|+q_0)}{|c+d|} + \frac{T^\gamma}{\Gamma(\gamma+1)}\left(\frac{|d|}{|c+d|}+1\right)\left(M_{\beta\gamma}(l_1\|y(t)\| + l_2R_s(\|x(t)\|+\|y(t)\|)) + l_0\right) \le (m_3 + M_3M_{\beta\gamma}l_2R_s)\|x(t)\| + M_3m_4\|y(t)\| + M_3l_0 + \frac{Tq_0}{|c+d|}$$

which implies

$$\|x(t)\| + \|y(t)\| \le (M_1m_2 + m_3 + M_3M_{\beta\gamma}l_2R_s)\|x(t)\| + (M_3m_4 + m_1 + M_1M_{\beta\alpha}k_2K_s)\|y(t)\| + M_1k_0 + M_3l_0 + \frac{Tp_0}{|a+b|} + \frac{Tq_0}{|c+d|}$$

Consequently

$$\|x(t)\| + \|y(t)\| \le \frac{T\left(\frac{p_0}{|a+b|} + \frac{q_0}{|c+d|}\right) + M_1k_0 + M_3l_0}{M_s}$$

which proves that $\xi(T_{fg})$ is bounded, thus by lemma 2.4, the operator T_{fg} has at least one fixed point. Hence the boundary value problem (1.1) and (1.2) has at least one solution.

Theorem 3.2 Assume that the functions $f, g \in C([0,T]\times\Omega\times D_1, D_2, R)$ be continuous functions and satisfy H_1, H_2 *and* H_3. In addition assume that $(m_1 + m_2M_1) + (m_3 + m_4M_3) < 1$

where

$$\left.\begin{aligned} m_1 = \frac{Tp_1}{|a+b|},\ m_2 = M_{\beta\alpha}(k_1 + k_2K_s),\ m_3 = \frac{Tq_1}{|c+d|}, \\ m_4 = M_{\beta\gamma}(l_1 + l_2R_s),\ M_1 = \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\frac{|b|}{|a+b|}+1\right) \\ and\ \ M_3 = \frac{T^\gamma}{\Gamma(\gamma+1)}\left(\frac{|d|}{|c+d|}+1\right) \end{aligned}\right\} \quad \ldots (2.15)$$

Then the fractional integro-differential equations (1.1) with boundary conditions (1.2) has a unique solution.

Proof. Since H_1, H_2 *and* H_3 are satisfy and we have

$$r \geq \frac{\frac{Tp_0}{|a+b|} + M_1 k_0 + \frac{Tq_0}{|c+d|} + M_3 l_0}{1 - [M_1 m_2 + m_1 + M_3 m_4 + m_3]}$$

First, we show that $T_{fg} B_r \subset B_r$, where

$$B_r = \{ (x,y) \in X \times Y : \|(x,y)\| \leq r)\}$$

For $(x,y) \in B_r$, we have

$$|T_1(x,y)(t)| \leq \frac{1}{|a+b|}\int_0^T |h_1(y(s))|ds + \frac{b}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\left|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu) - y(\mu))d\mu)\right| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu) - y(\mu))d\mu)\right| ds$$

$$\leq \frac{1}{|a+b|}\int_0^T |h_1(y(s))|ds + \frac{b}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} \left|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu)-y(\mu))d\mu) - f(s,\beta(\tau,\alpha),0,0) + f(s,\beta(\tau,\alpha),0,0)\right| ds + \frac{1}{\Gamma(\alpha)}$$

$$\int_0^t (t-s)^{\alpha-1}\left|f(s,\beta(\tau,\alpha),x(s),\int_0^{\omega(s)} K(s,\mu)(x(\mu) - y(\mu))d\mu) - f(s,\beta(\tau,\alpha),0,0) + f(s,\beta(\tau,\alpha),0,0)\right| ds$$

So that

$$|T_1(x,y)(t)| \leq \frac{T}{|a+b|}(p_1\|y(t)\| + p_0) + \frac{|b|T^\alpha}{|a+b|\Gamma(\alpha+1)} \times \left(M_{\beta\alpha}\left(k_1\|x(t)\| + k_2 K_s(\|x(t)\| + \|y(t)\|)\right) + k_0\right) + \frac{T^\alpha}{\Gamma(\alpha+1)} \times \left(M_{\beta\alpha}\left(k_1\|x(t)\| + k_2 K_s(\|x(t)\| + \|y(t)\|)\right) + k_0\right)$$

From (2.15), we can get

$$|T_1(x,y)(t)| \leq \left(\frac{Tp_1}{|a+b|}r + \frac{Tp_0}{|a+b|}\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(\frac{|b|}{|a+b|}+1\right)\left(M_{\beta\alpha}(k_1 + k_2 K_s)r + k_0\right)$$

$$\leq \left(m_1 r + \frac{Tp_0}{|a+b|}\right) + M_1(m_2 r + k_0)$$

In the same way, we can obtain that

$$|T_2(x,y)(t)| \leq \left(\frac{Tq_1}{|a+b|}r + \frac{Tq_0}{|c+d|}\right) + \frac{T^\gamma}{\Gamma(\gamma+1)}\left(\frac{|d|}{|c+d|}+1\right)\left(M_{\beta\gamma}(l_1 + l_2 R_s)r + l_0\right)$$

$$\leq \left(m_3 r + \frac{Tq_0}{|c+d|}\right) + M_3(m_4 r + l_0)$$

Consequently,

$$\left|T_{fg}(x,y)(t)\right| \leq \left(m_1 r + \frac{Tp_0}{|a+b|}\right) + M_1(m_2 r + k_0) + \left(m_3 r + \frac{Tq_0}{|c+d|}\right) + M_3(m_4 r + l_0) \leq r$$

Now, from $(x_1,y_1),(x_2,y_2) \in X \times Y$ and for any $t \in [0,T]$, we have

$$\left\|T_1(x_2,y_2)_{(t)} - T_1(x_1,y_1)_{(t)}\right\| \leq \frac{1}{|a+b|}\int_0^T |h_1(y_2(s)) - h_1(y_1(s))|ds + \frac{b}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|f(s,\beta(\tau,\alpha),x_2(s),\int_0^{\omega(s)} K(s,\mu)(x_2(\mu) - y_2(\mu))d\mu) - f(s,\beta(\tau,\alpha),x_1(s),\int_0^{\omega(s)} K(s,\mu)(x_1(\mu) - y_1(\mu))d\mu)\right| ds +$$

$$\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,\beta(\tau,\alpha),x_2(s),\int_0^{\omega(s)} K(s,\mu)(x_2(\mu) - y_2(\mu))d\mu) - f(s,\beta(\tau,\alpha),x_1(s),\int_0^{\omega(s)} K(s,\mu)(x_1(\mu) - y_1(\mu))d\mu)\Big| ds$$

$$\leq \frac{Tp_1}{|a+b|}\|y_2(t) - y_1(t)\| + \frac{|b|}{|a+b|}\frac{M_{\beta\alpha}T^\alpha}{\Gamma(\alpha+1)}(k_1\|x_2(t) - x_1(t)\| + k_2 K_s(\|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|) + \frac{M_{\beta\alpha}T^\alpha}{\Gamma(\alpha+1)}(k_1\|x_2(t) - x_1(t)\| + k_2 K_s(\|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|))$$

$$\leq \frac{Tp_1}{|a+b|}\|y_2(t) - y_1(t)\| + \frac{M_{\beta\alpha}T^\alpha}{\Gamma(\alpha+1)}\left(\frac{|b|}{|a+b|}+1\right)((k_1 + k_2 K_s)\|x_2(t) - x_1(t)\| + k_2 K_s\|y_2(t) - y_1(t)\|$$

From (2.15), we find that

$$\left\|T_1(x_2,y_2)_{(t)} - T_1(x_1,y_1)_{(t)}\right\| \leq M_1 m_2\|x_2(t) - x_1(t)\| + (m_1 + M_1 k_2 K_s)\|y_2(t) - y_1(t)\| \quad \ldots (2.16)$$

Similarly, we get

$$\left\|T_2(x_2,y_2)_{(t)} - T_2(x_1,y_1)_{(t)}\right\| \leq (\frac{Tq_1}{|c+d|} + \frac{M_{\beta\gamma}T^\gamma}{\Gamma(\gamma+1)}\left(\frac{|d|}{|c+d|}+1\right) l_2 R_s)\|x_2(t) - x_1(t)\| + \frac{M_{\beta\gamma}T^\gamma}{\Gamma(\gamma+1)}\left(\frac{|d|}{|c+d|}+1\right)(l_1 + l_2 R_s)\|y_2(t) - y_1(t)\|$$

$$\leq (m_3 + M_3 l_2 R_s)\|x_2(t) - x_1(t)\| + M_3 m_4\|y_2(t) - y_1(t)\| \quad \ldots (2.17)$$

It follows from (2.16) and (2.17) that

$$\|T_{fg}(x_2,y_2)(t) - T_{fg}(x_1,y_1)(t)\| \le [m_1 + M_1(k_2K_s + m_2) + m_3 + M_3(l_2R_s + m_4)] \times \|x_2(t) - x_1(t)\| + \|y_2(t) - y_1(t)\|) \quad \dots (2.18)$$

Since

$$m_1 + M_1(k_2K_s + m_2) + m_3 + M_3(l_2R_s + m_4) < 1$$

Therefore, T_{fg} is contraction operator, so by Banach fixed point, which is the unique solution of the boundary value problems (1.1) and (1.2). This completes the proof.

4. EXAMPLE

Consider the following system of fractional integro-differential equation

$$\left.\begin{aligned} {}^cD_{0+}^{0.5}x(t) &= \frac{1}{100\beta(\tau,\alpha)}(e^t + \sin(x(t)), \\ &+ \int_0^t (t+s)^{1/2}(x(s) - y(s))ds \quad t \in [0,2], \\ {}^cD_{0+}^{0.5}x(t) &= \frac{1}{2\beta(\tau,\gamma)}(\frac{2}{15} + \frac{|y(t)|}{9e^t(1+|y(t)|)}, \\ &+ \frac{1}{50}\int_0^{t^2} (t+s)^{1/3}(x(s) - y(s))ds \quad t \in [0,2] \end{aligned}\right\} \dots (4.1)$$

with non-separated integral coupled boundary condition

$$\left.\begin{aligned} 20x(0) + x(2) &= \int_0^2 \frac{1}{5(1+|y(s)|)}ds, \\ 30y(0) + y(2) &= \int_0^2 \frac{1}{(10+|x(s)|)}ds, \end{aligned}\right\} \dots (4.2)$$

where

$$\beta(\tau,\alpha) = \beta(\tau,\gamma) = \int_0^\infty \frac{s^{\tau-1}}{(1+s)^{\tau+0.5}}ds$$

means that $M_{\beta\alpha} = \max_{\tau\in(0,2]} \frac{1}{\beta(\tau,\alpha)} = 0.75 = M_{\beta\gamma}$

Here $T = 2$, $\alpha = \gamma = 0.5$, $a = 20$, $c = 30$, $b = d = 1$, $\omega(t) = t$, $\varphi(t) = t^2$,

$K(t,s) = (t+s)^{\frac{1}{2}}$, $R(t,s) = (t+s)^{\frac{1}{3}}$,

$h_1(y(t)) = \frac{1}{5(1+|y(t)|)}$, $h_2(x(t)) = \frac{1}{(10+|x(s)|)}$

we obtain that $K_s = 5.3333$, $R_s = 12$,
$k_1 = k_2 = 0.01$, $l_1 = 0.0556$, $l_2 = 0.01$,
$k_0 = 0.0554$, $l_0 = 0.05$, $p_0 = p_1 = 0.2$,
$q_0 = 0.1$, $q_1 = 0.01$, $m_1 = 0.019$,
$m_2 = 0.0475$, $m_3 = 0.00065$ $m_4 = 0.1317$,

$M_1 = 1.6718$, $M_3 = 1.6472$

Therefore

$$m_1 + M_1(k_2K_s + m_2) + m_3 + M_3(l_2R_s + m_4) = 0.6028 < 1$$

By Theorem 3.2, the coupled boundary value problem (4.1) and (4.2) has at least one solution.

5. CONCLUSION

In this paper, we have investigated the existence results for a system of nonlinear fractional integro-differential equations with coupled non-separated integral boundary conditions (1.1.) and (1.2) by using the Leray–Schauder fixed point theorem and uniqueness results for that system by using the Banach contraction principle and. Finally, we give example to demonstrate our results.

REFERENCES

[1] B. Ahmad, J. J.Nieto,**A class of differential equations of fractional order with multi-point boundary conditions**. Georgian Math. J. 2014, 21, 243–248.
https://doi.org/10.1515/gmj-2014-0025
[2] B.Ahmad, J. J. Nieto,**Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions**. Comput.Math. Appl. 58, 1838-1843 (2009)
https://doi.org/10.1016/j.camwa.2009.07.091
[3] M.Benchohra,A.Cabada, D.Seba, **An existence result for nonlinear fractional differential equations on Banach spaces**.Bound.Value Probl. 2009.
https://doi.org/10.1155/2009/628916
[4] R. N.Butris, A. Sh. Rafeeq and Faris, H. S., **Existence, uniqueness and stability of periodic solution for nonlinear system of integro-differential equations**.science Journal of University of Zakho Vol. 5.No. 1 pp. 120-127, (2017).
https://doi.org/10.25271/2017.5.1.312
[5]R. N. Butris, R. F. Taher**. Periodic solution of integro-differential equations depended on special functions withsingular kernels and boundary integral conditions.** International Journal of Advanced Trends in Computer Science and Engineering.Vol. 8, No.4, (2019)
https://doi.org/10.30534/ijatcse/2019/73842019
[6] AGranas, J. Dugundji: **Fixed Point Theory**. Springer, New York (2005)
[7] S. B. Hadid, **Local and global existence theorems on differential equations of non-integer order**.*Journal of Fractional Calculus*, vol. 7, pp. 101–105, 1995.
[8] J. Henderson,R.Luca,;Tudorache, A. **On a system of fractional equations with coupled integral boundary conditions**. Fract. Calc. Appl. Anal. 2015, 18, 361–386.
https://doi.org/10.1515/fca-2015-0024

[9] A. A. Kilbas,H. MSrivastava, J. J.Trujillo, **Theory and Applications of Fractional Differential Equations**, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)

[10] V. Lakshmikantham, S. Leela, Vasundhara Devi, J: **Theory of Fractional Dynamic Systems**. Cambridge Academic Publishers, Cambridge (2009)

[11] S. M. Momani, .**Local and global existence theorems on fractional integro-differential equations**,*Journal of Fractional Calculus*, vol. 18, pp. 81–86, 2000.

[12] Ntouyas and Obaid: **A coupled system of fractional differential equations with nonlocal integral boundary conditions**. *Advances in Difference Equations* 2012 2012:130.
https://doi.org/10.1186/1687-1847-2012-130

[13] J. Sabatier, OP. Agrawal, Machado, JAT (eds.): **Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering**. Springer, Dordrecht (2007)
https://doi.org/10.1007/978-1-4020-6042-7

[14] X. Su,**Boundary value problem for a coupled system of nonlinear fractional differential equations**. Appl. Math. Lett. 22, 64-69 (2009)
https://doi.org/10.1016/j.aml.2008.03.001

[15] J. Wang, H. Xiang, , Z. Liu, : **Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations.** Int. J. Differ. Equ. 2010, Article ID 186928 (2010)
https://doi.org/10.1155/2010/186928