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Periodic Solution of Integro-Differential Equations Depended on Special Functions with Singular Kernels and Boundary Integral Conditions

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ABSTRACT

This article investigates periodic solutions for new nonlinear of integro-differential equations depended on special functions with singular kernels and boundary integral conditions by using the numerical analytic method which was introduced by Samoilenko method. Theorems on existence and uniqueness of a periodic solution are established under some necessary and sufficient conditions on closed and bounded domains.

Key words: Numerical-analytic method, periodic solutions, integro-differential equations, singular kernels , boundary integral conditions**,** stability solution.

1. INTRODUCTION

A boundary value problem consists of a differential equation on a given interval and an explicit condition that the solution must satisfy at one or several points. The simplest instance of such explicit conditions is when they are all specified at one initial point. The solution of differential equations may be generally specified at more than one point. Often there are two points, which correspond physically to the boundaries of some region, so that it is a two–point boundary value problem [11]. The study of periodic solutions for non-linear system of differential equations with boundary conditions and boundary integral conditions is a very important branch in the differential equation theory [6,7,8,13,20]. Many results about the existence and approximation of periodic solutions for system of non–linear differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko [3,4] which had been later applied in many studies [1,9,14,15,18].

Samoilenko [5] has used the numerical-analytic methods of periodic solutions for ordinary differential equation with boundary and boundary integral conditions which has the form:

$$\frac{dx}{dt} = f(t,x)$$

$$\int_0^T x(t)\,dt = d\,, \quad d \in R^n,$$

where $x \in D$, D is closed and bounded subset of R^n, the vector function $f(t,x)$ is defined on the domain :

$$(t,x) \in R^1 \times D = (-\infty,\infty) \times D$$

which is continuous in t, x and periodic in t of period T.

Lemma 1.1 [4]. Let $f(t)$ be a vector function which is defined in the interval $0 \le t \le T$, then :

$$\left|\int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds\right| \le \alpha(t)M\,,$$

where $M = \max_{t\in[0,T]}|f(t)|$ and $\alpha(t) = 2t(1-\frac{t}{T})$.

In this work, we investigate the existence and approximation of periodic solution for non-linear integro-differential equations with boundary integral conditions.

$$\left.\begin{aligned} \frac{dx}{dt} &= f(t,\gamma(t,\alpha),\beta(t,\alpha),x(t),\mu,\varepsilon,u) \\ &\int_0^T x(t)dt = d_1 \\ \frac{dx}{dt} &= g(t,\gamma(t,\alpha),\beta(t,\alpha),y(t),\omega,\eta,n) \\ &\int_0^T y(t)dt = d_2 \end{aligned}\right\} \qquad (1)$$

where

$$\mu = \int_a^b \gamma(\tau,\alpha)x(\tau)d\tau\,, \varepsilon = \int_a^b \beta(\tau,\alpha)x(\tau)d\tau,$$

$$\omega = \int_c^d \gamma(\tau,\alpha)y(\tau)d\tau\,, \eta = \int_c^d \beta(\tau,\alpha)y(\tau)d\tau,$$

$$u = \int_{-\infty}^t R(t,\tau)\big(x(\tau)-y(\tau)\big)d\tau,$$

$$v = \int_{-\infty}^t G(t,\tau)\,\big(x(\tau)-y(\tau)\big)d\tau$$

Where $x \in D_1 \subset R^n, y \in D_2 \subset R^n, D_1$ and D_2 are compact domains.

Let the vector functions $(t,\gamma(t,\alpha),\ \beta(t,\alpha),x,\mu,\varepsilon,u)$ and $g(t,\gamma(t,\alpha),\beta(t,\alpha),y,\omega,\eta,v)$ are defined and continuous on the domain :-

$$\left.\begin{aligned} &(t,\gamma(t,\alpha),\beta(t,\alpha),x,\mu,u) \in R^1 \times G_1 = \\ &\quad(-\infty,\infty)\times D \times D_1 \times D_\mu \times D_\varepsilon \times D_u \\ &(t,\gamma(t,\alpha),\beta(t,\alpha),y,\omega,v) \in R^1 \times G_2 = \\ &\quad(-\infty,\infty)\times D \times D_2 \times D_\omega \times D_\eta \times D_v \end{aligned}\right\} \qquad (2)$$

Also $D_\mu, D_\varepsilon, D_\omega, D_\eta, D_u$ and D_v are bounded domains subset of Euclidean space R^m.

where $D = [\tau, \tau + T] \times (0,1]$.

and periodic in t of period T.

Suppose that the functions $f(t,\gamma(t,\alpha), \beta(t,\alpha), x, \mu, \varepsilon, u)$ and $g(t,\gamma(t,\alpha),\beta(t,\alpha), y, \omega, \eta, v)$ satisfy the following inequalities

$$\left.\begin{array}{c} \| f(t,\gamma(t,\alpha),\beta(t,\alpha),x,\mu,\varepsilon,u) \| \le \|\gamma(t,\alpha)\|\|\beta(t,\alpha)\| \\ \| f(t,x,\mu,\varepsilon,u)\| \le M_\gamma M_\beta M \\ \| g(t,\gamma(t,\alpha),\beta(t,\alpha),y,\omega,\eta,v) \| \le \|\gamma(t,\alpha)\|\|\beta(t,\alpha)\| \\ \| g(t,y,\omega,\eta,v) \| \le N_\gamma N_\beta N \end{array}\right\} \quad (3)$$

$$\| f(t,\gamma(t,\alpha),\beta(t,\alpha),x_1,\mu_1,\varepsilon_1,u_1) - f(t,\gamma(t,\alpha),\beta(t,\alpha),x_2,\mu_2,\varepsilon_2,u_2) \|$$
$$\le M_\gamma M_\beta (K_1 \| x_1 - x_2 \| + K_2 \| \mu_1 - \mu_2 \| + K_3 \| \varepsilon_1 - \varepsilon_2 \| + K_4 \| u_1 - u_2 \| \quad (4)$$

$$\| g(t,\gamma(t,\alpha),\beta(t,\alpha),y_1,\omega_1,\eta_1,v_1) - g(t,\gamma(t,\alpha),\beta(t,\alpha),y_2,\omega_2,\eta_2,v_2) \|$$
$$\le N_\gamma N_\beta (L_1 \| y_1 - y_2 \| + L_2 \| \omega_1 - \omega_2 \| + L_3 \| \eta_1 - \eta_2 \| + L_4 \| v_1 - v_2 \| \quad (5)$$

for all $t \in R^1, x, x_1, x_2 \in D_1, y, y_1, y_2 \in D_2, \mu, \varepsilon, \omega, \eta, u$ and v are belong to $D_\mu, D_\varepsilon, D_\omega, D_\eta, D_u$ and D_v respectively, where $M, N, K_1, K_2, K_3, K_4, L_1, L_2, L_3$ and L_4 are positive constants. Also $\gamma(t,\alpha)$ and $\beta(t,\alpha)$ are special functions (Gamma and Beta functions) provided that $\gamma(t,\alpha) = \gamma(t+T,\alpha)$ and $\beta(t,\alpha) = \beta(t+T,\alpha)$ the singular kernels $R(t,\tau)$ and $G(t,\tau)$ satisfying the following conditions :-

$$\left.\begin{array}{l} \| R(t,\tau) \| \le h e^{-\alpha(t-\tau)} \\ \| G(t,\tau) \| \le \sigma e^{-\delta(t-\tau)} \end{array}\right\} \quad (6)$$

where $-\infty < 0 \le \tau \le t \le \tau + T, \alpha$ and δ are positive constants.

Now, we defined the non-empty sets as follows :-

$$\left.\begin{array}{l} D_{1I} = G_1 - (\frac{T}{2} M_\gamma M_\beta M + \frac{2}{T-2\tau} q(x_0)) \\ D_{2I} = G_2 - (\frac{T}{2} N_\gamma N_\beta N + \frac{2}{T-2\tau} p(y_0)) \end{array}\right\} \quad (7)$$

where

$$q(x_0) = \|d_1 - x_0 T\| + \left(\frac{T^2}{3} - 2\tau^2\right) M_\gamma M_\beta M$$

and

$$p(y_0) = \|d_2 - y_0 T\| + (\frac{T^2}{3} - 2\tau^2) N_\gamma N_\beta N$$

Furthermore, we suppose that the largest eigen-value of the matrix

$$Q = \begin{pmatrix} \alpha(t) M_\gamma M_\beta F_1 r_1 & \alpha(t) M_\gamma M_\beta F_2 r_1 \\ \alpha(t) N_\gamma N_\beta F_3 r_1 & \alpha(t) N_\gamma N_\beta F_4 r_1 \end{pmatrix}$$ less than one ,i.e.

$$q_{max}(Q) = \frac{\delta_1 + \sqrt{\delta_1^2 + 4(\delta_2 - \delta_3)}}{2} < 1 \quad (8)$$

where $\delta_1 = \frac{T}{2} M_\gamma M_\beta F_1 r_1 + \frac{T}{2} N_\gamma N_\beta F_4 r_1$,

$\delta_2 = \left(\frac{T}{2} M_\gamma M_\beta F_2 r_1\right)\left(\frac{T}{2} N_\gamma N_\beta F_3 r_1\right)$, $\delta_3 = \left(\frac{T}{2} M_\gamma M_\beta F_1 r_1\right)$ $\left(\frac{T}{2} N_\gamma N_\beta F_4 r_1\right)$, $F_1 = K_1 + K_2 M_\gamma (b-a) + K_3 M_\beta (b-a) + \frac{h}{\alpha} K_4$, $F_2 = \frac{h}{\alpha} K_4$, $F_3 = \frac{\sigma}{\delta} L_4$, $F_4 = L_1 + L_2 N_\gamma (d-c) + L_3 N_\beta (d-c) + \frac{\sigma}{\delta} L_4$, $r_1 = 1 + \frac{2T}{T-2\tau}$

2.APPROXIMATION OF PERIODIC SOLUTION (1)

In this section, we study the periodic approximation solution of (1) will be introduced by the following theorem :-

Theorem 2.1. If the system (1) satisfy the inequalities (3) to (6) and the conditions (7) and (8) has periodic solutions $x = x(t, x_0, y_0)$ and $y = y(t, x_0, y_0)$, then the sequence of functions :-

$$x_{m+1}(t,x_0,y_0) = x_0 + \int_\tau^t \Big[f(\tau,\gamma(\tau,\alpha),,\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\mu_m,\varepsilon_m,u_m) - \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\mu_m,\varepsilon_m,u_m) \Big] d\tau + (t-\tau)\rho \quad (9)$$

with
$x_0(t,x_0,y_0) = x_0$

$$\rho = \frac{2}{T(T-2\tau)}[\, d_1 - x_0 T - \int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha), x_m(t,x_0,y_0),\mu_m,\varepsilon_m,u_m) dt \,]$$

and

$$Lf(t,\gamma(t,\alpha),\beta(t,\alpha),x_m(t,x_0,y_0),\mu_m,\varepsilon_m,u_m)$$
$$= \int_\tau^t [\, f(\tau,\gamma(\tau,\alpha), \beta(\tau,\alpha), x_m(\tau,x_0,y_0), \mu_m,\varepsilon_m,u_m) - \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\mu_m,\varepsilon_m,u_m) d\tau]$$

and

$$y_{m+1}(t,x_0,y_0) = y_0 + \int_\tau^t \Big[g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha), y_m(\tau,x_0,y_0),\omega_m,\eta_m v_m) - \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y_m(\tau,x_0,y_0),\omega_m,\eta_m v_m) \Big] d\tau + (t-\tau)\vartheta \quad (10)$$

with
$y_0(t,x_0,y_0)=y_0$

where

$$\vartheta=\frac{2}{T(T-2\tau)}[d_2-y_0T-\int_0^T Lg(t,\gamma(t,\alpha),\beta(t,\alpha),$$
$$y_m(t,x_0,y_0),\omega_m,\eta_m,v_m)dt]$$

and

$$Lg(t,\gamma(t,\alpha),\beta(t,\alpha),y_m(t,x_0,y_0),\omega_m,\eta_m,,v_m)$$
$$=\int_\tau^t[\,g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha)\,,y_m(\tau,x_0,y_0)\,\,,\omega_m,\eta_m,v_m)$$
$$-\frac{1}{T}\int_\tau^{\tau+T}g(\tau,\gamma(\tau,\alpha)\beta(\tau,\alpha)\,,y_m(\tau,x_0,y_0)\,,\omega_m,\eta_m,v_m)]d\tau$$

where

$$\mu_m=\int_a^b\gamma(\tau,\alpha)x_m(\tau,x_0,y_0)d\tau,\varepsilon_m=\int_a^b\beta(\tau,\alpha)x_m(\tau,x_0,y_0)d\tau,$$
$$\omega_m=\int_c^d\gamma(\tau,\alpha)y_m(\tau,x_0,y_0)d\tau,\eta_m=\int_c^d\beta(\tau,\alpha)y_m(\tau,x_0,y_0)d\tau$$
$$u_m=\int_{-\infty}^t R(t,\tau)(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0))d\tau$$
$$v_m=\int_{-\infty}^t G(t,\tau)(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0))d\tau$$

are periodic in t of period T, and uniformly converges as $m\to\infty$ on the domain :-

$$(t,x_0,y_0)\in[\tau,\tau+T]\times D_{1f}\times D_{2f} \tag{11}$$

to the limit function $\begin{pmatrix}x\,(t,x_0,y_0)\\ y(t,x_0,y_0)\end{pmatrix}$ defined on the domain (11) which is periodic in t of period T and satisfying the following integral equations :-

$$x(t,x_0,y_0)=x_0+\int_\tau^t\Big[f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,$$
$$\varepsilon,u)\;-\frac{1}{T}\int_\tau^{\tau+T}f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu\,,\varepsilon,u)\Big]d\tau$$
$$+(t-\tau)\rho \tag{12}$$

and

$$y(t,x_0,y_0)=y_0+\int_\tau^t\Big[g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,$$
$$\eta,v)-\frac{1}{T}\int_\tau^{\tau+T}g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,\eta,v)d\tau\Big]d\tau$$
$$+(t-\tau)\vartheta \tag{13}$$

which are unique solutions on the domain (11), provided that

$$\begin{pmatrix}\|\,x(t,x_0,y_0)-x_0\,\|\\ \\ \|\,y(t,x_0,y_0)-y_0\,\|\end{pmatrix}\le\begin{pmatrix}\alpha(t)M_\gamma M_\beta M+\dfrac{2}{T-2\tau}q(x_0)\\ \\ \alpha(t)N_\gamma N_\beta N+\dfrac{2}{T-2\tau}p(y_0)\end{pmatrix} \tag{14}$$

and

$$\begin{pmatrix}\|\,x(t,x_0,y_0)-x_m(t,x_0,y_0)\,\|\\ \|\,y(t,x_0,y_0)-y_m(t,x_0,y_0)\,\|\end{pmatrix}\le\;Q^m(E-Q)^{-1}\psi_1 \tag{15}$$

for all $m\ge 0,x_0\in D_{1f},y_0\in D_{2f}$ and $t\in R^1$, where

$$\psi_1=\begin{pmatrix}\alpha(t)M_\gamma M_\beta M+\frac{2}{T-2\tau}q(x_0)\\ \\ \alpha(t)N_\gamma N_\beta N+\frac{2}{T-2\tau}p(y_0)\end{pmatrix},$$ E is identity matrix.

Proof. Consider the sequence of functions $x_1(t,x_0,y_0),x_2(t,x_0,y_0),\dots,x_m(t,x_0,y_0)\,,\dots$ and $y_1(t,x_0,y_0)$ $,y_2(t,x_0,y_0),\dots,y_m(t,x_0,y_0)\;,\dots,$ defined by the recurrences relation (9) and (10) Each of these functions of sequence defined and continuous in the domain (2) and periodic in t of period T.

Now, by the Lemma 1.1 and using the sequence of functions (9) and(10) when $m=0$, we get

$$\|x_1(t,x_0,y_0)-x_0\|=\Bigg\|\int_\tau^t\Big[f(\tau,\gamma(\tau,\alpha),,\beta(\tau,\alpha),x_0,$$
$$,\int_a^b\gamma(\tau,\alpha)\;x_0d\tau,\;\int_a^b\beta(\tau,\alpha)\;x_0d\tau,\int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)$$
$$-\frac{1}{T}\int_\tau^{\tau+T}f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x_0,\int_a^b\gamma(\tau,\alpha)\;x_0d\tau,\int_a^b\beta(\tau,\alpha)$$
$$x_0d\tau\,,\int_{-\infty}^t R(t,\tau)\,(x_0-y_0)d\tau)d\tau\Big]d\tau+\;\frac{2(t-\tau)}{T(T-2\tau)}[\,d_1$$
$$-x_0T-\int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha),x_0,\mu,\varepsilon,u)dt\,]\Bigg\|$$

$$\le\left(1-\frac{t-\tau}{T}\right)\int_\tau^t\|\gamma(\tau,\alpha)\|\;\;\|\beta(\tau,\alpha)\|\Bigg\|f(\tau,x_0,\int_a^b\gamma(\tau,\alpha)$$
$$x_0d\tau,\int_a^b\beta(\tau,\alpha)\;x_0d\tau,\int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)\Bigg\|d\tau$$
$$+\frac{t-\tau}{T}\int_t^{\tau+T}\|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|\Bigg\|f(\tau,x_0,\int_a^b\gamma(\tau,\alpha)\;x_0d\tau,$$

$$\int_a^b \beta(\tau,\alpha)\, x_0 d\tau\,, \int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)\Bigg\|\, d\tau$$

$$+\frac{2}{T-2\tau}[\, \|d_1 - x_0 T\| + \int_0^T \Bigg[\left(1-\frac{t-\tau}{T}\right)\int_\tau^t \|\gamma(\tau,\alpha)\|$$

$$\|\beta(\tau,\alpha)\| \Bigg\| f(\tau, x_0, \int_a^b \gamma(\tau,\alpha)\, x_0 d\tau, \int_a^b \beta(\tau,\alpha)\, x_0 d\tau,$$

$$\int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)\Bigg\|\, d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\, \|\beta(\tau,\alpha)\| \Bigg\| f(\tau, x_0\,, \int_a^b \gamma(\tau,\alpha)\, x_0 d\tau,$$

$$\int_a^b \beta(\tau,\alpha)\, x_0 d\tau, \int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)\Bigg\|\, d\tau)\Bigg] dt\,]$$

hence

$$\|x_1(t,x_0,y_0) - x_0\| \le \alpha(t) M_\gamma M_\beta M + \frac{2}{T-2\tau} q(x_0) \quad (16)$$

for all $x_1(t,x_0,y_0) \in G_1$, for all $t \in [\tau, \tau+T], x_0 \in D_{1f}$.

Thus, by mathematical induction, we can prove that

$$\|x_m(t,x_0,y_0) - x_0\| \le \alpha(t) M_\gamma M_\beta M + \frac{2}{T-2\tau} q(x_0) \quad (17)$$

Then $x_m(t,x_0,y_0) \in G_1$, for all $t \in [\tau, \tau+T], x_0 \in D_{1f}$.

similarly

$$\|y_1(t,x_0,y_0) - y_0\| \le \alpha(t) N_\gamma N_\beta N + \frac{2}{T-2\tau} p(y_0) \quad (18)$$

for all $y_1(t,x_0,y_0) \in G_2$, for all $t \in [\tau, \tau+T],\ y_0 \in D_{2f}$.

and

$$\|y_m(t,x_0,y_0) - y_0\| \le \alpha(t) N_\gamma N_\beta N + \frac{2}{T-2\tau} p(y_0) \quad (19)$$

for all $y_m(t,x_0,y_0) \in G_2$, for all $t \in [\tau, \tau+T],\ y_0 \in D_{2f}$.

From the inequalities (18) and (19), we get (14).

Next, we shall prove that the sequence of functions (9) and (10) are convergent uniformly on the domain (11).

When $m = 1$ and by using Lemma 1.1 and the inequalities (4) and (5), we obtains

$$\|x_2(t,x_0,y_0) - x_1(t,x_0,y_0)\| = \Bigg\|\int_\tau^t \Bigg[[\, f(\tau, \gamma(\tau,\alpha), \beta(\tau,\alpha),$$

$$x_1(\tau,x_0,y_0), \int_a^b \gamma(\tau,\alpha) x_1(\tau,x_0,y_0) d\tau, \int_a^b \beta(\tau,\alpha)$$

$$x_1(\tau,x_0,y_0) d\tau, \int_{-\infty}^t R(t,\tau)\,(x_1(\tau,x_0,y_0) - y_1(\tau,x_0,y_0))d\tau)$$

$$-f(\tau, \gamma(\tau,\alpha), \beta(\tau,\alpha), x_0, \int_a^b \gamma(\tau,\alpha)\, x_0 d\tau, \int_a^b \beta(\tau,\alpha)\, x_0 d\tau$$

$$, \int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)\,]$$

$$-\frac{1}{T}\int_\tau^{\tau+T} [\, f(\tau, \gamma(\tau,\alpha), \beta(\tau,\alpha), x_1(\tau,x_0,y_0)\,, \int_a^b \gamma(\tau,\alpha)$$

$$x_1(\tau,x_0,y_0) d\tau, \int_a^b \beta(\tau,\alpha) x_1(\tau,x_0,y_0) d\tau, \int_{-\infty}^t R(t,\tau)$$

$$(x_1(\tau,x_0,y_0) - y_1(\tau,x_0,y_0))d\tau) - f(\tau, \gamma(\tau,\alpha), \beta(\tau,\alpha), x_0$$

$$, \int_a^b \gamma(\tau,\alpha)\, x_0 d\tau, \int_a^b \beta(\tau,\alpha)\, x_0 d\tau, \int_{-\infty}^t R(t,\tau)(x_0-y_0)d\tau)]d\tau\Bigg] d\tau$$

$$+\frac{2(t-\tau)}{T(T-2\tau)}\int_0^T \Bigg[Lf(\, t, \gamma(t,\alpha), \beta(t,\alpha), x_1(t,x_0,y_0), \mu_1, \varepsilon_1, u_1)$$

$$-Lf(t, \gamma(t,\alpha), \beta(t,\alpha), x_0, \mu_0, \varepsilon_0, u_0)\Bigg] dt\Bigg\|$$

$$\le \left(1-\frac{t-\tau}{T}\right)\int_\tau^t \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Bigg[(K_1 + K_2 M_\gamma (b-a)$$

$$+K_3 M_\beta (b-a) + \frac{h}{\alpha} K_4)\, \|x_1(\tau,x_0,y_0) - x_0\| + \frac{h}{\alpha} K_4$$

$$\|y_1(\tau,x_0,y_0) - y_0\|\Bigg] d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Bigg[(K_1 + K_2 M_\gamma (b-a)$$

$$+K_3 M_\beta (b-a) + \frac{h}{\alpha} K_4) \|x_1(\tau,x_0,y_0) - x_0\| + \frac{h}{\alpha} K_4$$

$$\|y_1(\tau,x_0,y_0) - y_0\|\Bigg] d\tau$$

$$+\frac{2}{T-2\tau}\int_0^T \Bigg[(1-\frac{t-\tau}{T})\int_\tau^t \|\gamma(\tau,\alpha)\|\, \|\beta(\tau,\alpha)\|$$

$$\Bigg[(K_1 + K_2 M_\gamma (b-a) + K_3 M_\beta (b-a) + \frac{h}{\alpha} K_4)$$

$$\|x_1(\tau,x_0,y_0) - x_0\| + \frac{h}{\alpha} K_4 \|y_1(\tau,x_0,y_0) - y_0\|\Bigg] d\tau$$

$$+\frac{t-\tau}{T}\int_{t}^{\tau+T}\|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|\Bigg[(K_1+K_2M_\gamma(b-a)$$

$$+K_3M_\beta(b-a)+\frac{h}{\alpha}K_4)\|x_1(\tau,x_0,y_0)-x_0\|+\frac{h}{\alpha}K_4$$

$$\|y_1(\tau,x_0,y_0)-y_0\|\Bigg]d\tau\Bigg]dt$$

therefore

$$\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\|\le\alpha(t)M_\gamma M_\beta F_1r_1$$

$$\|x_1(t,x_0,y_0)-x_0\|+\ \alpha(t)M_\gamma M_\beta F_2r_1\|y_1(t,x_0,y_0)-y_0\| \quad (20)$$

where

$$\mu_1=\int_a^b\gamma(\tau,\alpha)x_1(\tau,x_0,y_0)d\tau,\varepsilon_1=\int_a^b\beta(\tau,\alpha)x_1(\tau,x_0,y_0)d\tau$$

$$\omega_1=\int_c^d\gamma(\tau,\alpha)y_1(\tau,x_0,y_0)d\tau,\eta_1=\int_c^d\beta(\tau,\alpha)y_1(\tau,x_0,y_0)d\tau$$

$$u_1=\int_{-\infty}^{t}R(t,\tau)(x_1(\tau,x_0,y_0)-y_1(\tau,x_0,y_0))d\tau$$

$$v_1=\int_{-\infty}^{t}G(t,\tau)(x_1(\tau,x_0,y_0)-y_1(\tau,x_0,y_0))d\tau$$

$$\mu_0=\int_a^b\gamma(\tau,\alpha)x_0\,d\tau,\varepsilon_0=\int_a^b\beta(\tau,\alpha)x_0\,d\tau$$

$$\omega_0=\int_c^d\gamma(\tau,\alpha)y_0\,d\tau,\eta_0=\int_c^d\beta(\tau,\alpha)y_0\,d\tau,$$

$$u_0=\int_{-\infty}^{t}R(t,\tau)(x_0-y_0)d\tau,v_0=\int_{-\infty}^{t}G(t,\tau)(x_0-y_0)d\tau$$

Consequently, by mathematical induction we can prove that

$$\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\|\ \le\alpha(t)M_\gamma M_\beta F_1r_1$$

$$\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|+\ \alpha(t)M_\gamma M_\beta F_2r_1$$

$$\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\| \quad (21)$$

and

$$\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\|\le\alpha(t)N_\gamma N_\beta F_3r_1$$

$$\|x_1(t,x_0,y_0)-x_0\|+\ \alpha(t)N_\gamma N_\beta F_4r_1\|y_1(t,x_0,y_0)-y_0\| \quad (22)$$

Thus, by induction we can prove that

$$\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\le\alpha(t)N_\gamma N_\beta F_3r_1$$

$$\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|+\ \alpha(t)N_\gamma N_\beta F_4r_1$$

$$\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\| \quad (23)$$

rewrite (21) and (22) in a vector form, i.e.

$$\psi_{m+1}(t)\le Q(t)\psi_m(t) \quad (24)$$

where

$$\psi_{m+1}=\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\|\\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix}$$

$$\psi_m=\begin{pmatrix}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\|\\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}$$

and

$$Q(t)=\begin{pmatrix}\alpha(t)\,M_\gamma M_\beta F_1r_1 & \alpha(t)\,M_\gamma M_\beta F_2r_1\\ \alpha(t)\,N_\gamma N_\beta F_3r_1 & \alpha(t)\,N_\gamma N_\beta F_4r_1\end{pmatrix}$$

Now, we take the maximum value of the both sides of the inequality (24), we get

$$\psi_{m+1}\le Q\,\psi_m \quad (25)$$

where $Q=max_{t\in[\tau,\tau+T]}\,Q(t)$, we obtain

$$Q=\begin{pmatrix}\frac{T}{2}M_\gamma M_\beta F_1r_1 & \frac{T}{2}M_\gamma M_\beta F_2r_1\\ \frac{T}{2}N_\gamma N_\beta F_3r_1 & \frac{T}{2}N_\gamma N_\beta F_4r_1\end{pmatrix}$$

and by repetition of (25), we find that

$$\psi_{m+1}\le Q^m\psi_1 \quad (26)$$

and also we get

$$\sum_{i=1}^{m}\psi_i\le\sum_{i=1}^{m}Q^{i-1}\psi_1 \quad (27)$$

By condition (8) then the series (27) is uniformly convergent that is

$$\lim_{m\to\infty}\sum_{i=1}^{m}Q^{i-1}\psi_1\le\sum_{i=1}^{\infty}Q^{i-1}\psi_1=(E-Q)^{-1}\psi_1 \quad (28)$$

Let

$$\lim_{m\to\infty}\begin{pmatrix}x_m(t,x_0,y_0)\\ y_m(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x\,(t,x_0,y_0)\\ y(t,x_0,y_0)\end{pmatrix} \quad (29)$$

Finally, we show that $\begin{pmatrix}x\,(t,x_0,y_0)\\ y(t,x_0,y_0)\end{pmatrix}$ is a unique solution of the system (1). Assume that $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$ is another solution for the system (1) i.e.

$$\dot{x}(t,x_0,y_0)=x_0+\int_{\tau}^{t}\Bigg[f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),\dot{x}(\tau,x_0,y_0),\mu,\varepsilon,u)$$

$$-\frac{1}{T}\int_{\tau}^{\tau+T}f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),\dot{x}(\tau,x_0,y_0),\mu,\varepsilon,u)\Bigg]d\tau$$

$$+(t-\tau)\rho \quad (30)$$

and

$$\dot{y}(t,x_0,y_0)=y_0+\int_{\tau}^{t}\Bigg[g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),\dot{y}(\tau,x_0,y_0),\omega,$$

$$\eta, v) \; - \frac{1}{T}\int_{\tau}^{\tau+T} g(\tau, \gamma(\tau,\alpha), \beta(\tau,\alpha), \dot{y}(\tau, x_0, y_0), \omega, \eta, v)\Bigg] d\tau$$

$$+(t-\tau)\vartheta \tag{31}$$

where

$$\mu = \int_a^b \gamma(\tau,\alpha)\dot{x}(\tau,x_0,y_0)d\tau, \varepsilon = \int_a^b \beta(\tau,\alpha)\dot{x}(\tau,x_0,y_0)d\tau,$$

$$\omega = \int_c^d \gamma(\tau,\alpha)\dot{y}(\tau,x_0,y_0)d\tau, \eta = \int_c^d \beta(\tau,\alpha)\dot{y}(\tau,x_0,y_0)d\tau,$$

$$u = \int_{-\infty}^t R(t,\tau)(\dot{x}(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0))d\tau,$$

$$v = \int_{-\infty}^t G(t,\tau)(\dot{x}(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0))d\tau$$

Now, we find the difference between them,

$$\| x(t,x_0,y_0) - \dot{x}(t,x_0,y_0) \| \le \left(1 - \frac{t-\tau}{T}\right)\int_{\tau}^{t} \|\gamma(\tau,\alpha)\|$$

$$\|\beta(\tau,\alpha)\| \Bigg[\;(K_1 + K_2M_\gamma(b-a) + K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)$$

$$\|x(\tau,x_0,y_0) - \dot{x}(\tau,x_0,y_0)\| + \frac{h}{\alpha}K_4$$

$$\|y(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0)\|\Bigg] d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|$$

$$\Bigg[(K_1 + K_2M_\gamma(b-a) + K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)$$

$$\|x(\tau,x_0,y_0) - \dot{x}(\tau,x_0,y_0)\| + \frac{h}{\alpha}K_4$$

$$\|y(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0)\|\Bigg] d\tau$$

$$+\frac{2}{T-2\tau}\int_0^T \Bigg[(1 - \frac{t-\tau}{T})\int_{\tau}^{t} \|\gamma(\tau,\alpha)\| \; \|\beta(\tau,\alpha)\|$$

$$\Bigg[(K_1 + K_2M_\gamma(b-a) + \; K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)$$

$$\|x(\tau,x_0,y_0) - \dot{x}(\tau,x_0,y_0)\| + \frac{h}{\alpha}K_4$$

$$\|y(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0)\|\Bigg] d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|\Bigg[(K_1 + K_2M_\gamma(b-a)$$

$$+ K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)\|x(\tau,x_0,y_0) - \dot{x}(\tau,x_0,y_0)\|$$

$$+\frac{h}{\alpha}K_4\|\|y(\tau,x_0,y_0) - \dot{y}(\tau,x_0,y_0)\|\|\Bigg] d\tau \Bigg] dt$$

so,

$$\| x(t,x_0,y_0) - \dot{x}(t,x_0,y_0) \| \le \alpha(t)M_\gamma M_\beta F_1 r_1$$

$$\|x(t,x_0,y_0) - \dot{x}(t,x_0,y_0)\| + \; \alpha(t)M_\gamma M_\beta F_2 r_1$$

$$\|y(t,x_0,y_0) - \dot{y}(t,x_0,y_0)\| \tag{32}$$

also, we find

$$\| y(t,x_0,y_0) - \dot{y}(t,x_0,y_0) \| \le \alpha(t)N_\gamma N_\beta F_3 r_1$$

$$\|x(t,x_0,y_0) - \dot{x}(t,x_0,y_0)\| + \; \alpha(t)N_\gamma N_\beta F_4 r_1$$

$$\|y(t,x_0,y_0) - \dot{y}(t,x_0,y_0)\| \tag{33}$$

Rewrite the inequalities (32) and (33) in a vector form, we get

$$\begin{pmatrix} \| x(t,x_0,y_0) - \dot{x}(t,x_0,y_0) \| \\ \| y(t,x_0,y_0) - \dot{y}(t,x_0,y_0) \| \end{pmatrix} \le Q \begin{pmatrix} \| x(t,x_0,y_0) - \dot{x}(t,x_0,y_0) \| \\ \| y(t,x_0,y_0) - \dot{y}(t,x_0,y_0) \| \end{pmatrix} \tag{34}$$

Now, by the condition (8).

$$\begin{pmatrix} \| x(t,x_0,y_0) - \dot{x}(t,x_0,y_0) \| \\ \| y(t,x_0,y_0) - \dot{y}(t,x_0,y_0) \| \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

that is

$$\begin{pmatrix} x\,(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix} = \begin{pmatrix} \dot{x}(t,x_0,y_0) \\ \dot{y}(t,x_0,y_0) \end{pmatrix}$$

this proves that the solution is a unique of the system (1) for all $t \in [\tau, \tau+T], x_0 \in D_{1f}, y_0 \in D_{2f}, \alpha(t) \le \frac{T}{2}$.

3. EXISTENCE OF PERIODIC SOLUTION OF THE SYSTEM (1)

The existence of periodic solution for the system (1) is uniquely connected with existence of zeros of the functions $\Delta_1^{**}(0,x_0,y_0)$ and $\Delta_2^{**}(0,x_0,y_0)$ which defined by :-

$$\Delta_1^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}[x_0T - d_1 + \int_0^T Lf\,(t,\gamma(t,\alpha),\beta(t,\alpha),x(t,x_0,y_0),\mu,\varepsilon,u)dt\,]$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,\varepsilon,u)d\tau \tag{35}$$

$$\Delta_2^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}[y_0T - d_2 + \int_0^T Lg\,(t,\gamma(t,\alpha),$$

$$\beta(t,\alpha), y(t,x_0,y_0), \omega, \eta, v)dt\,]$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,\eta,v)d\tau \qquad (36)$$

where $\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$ is the limiting function of the sequence of functions (9) and (10).

The equations (35) and (36) are approximately determined from the sequence of functions :-

$$\Delta_{1m}^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}[\,x_0T - d_1 + \int_0^T Lf(t,\gamma(t,\alpha),$$

$$\beta(t,\alpha), x_m(t,x_0,y_0), \mu_m, \varepsilon_m, u_m)\,dt\,]$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha)x_m(\tau,x_0,y_0),\mu_m,\varepsilon_m,u_m)d\tau \quad (37)$$

and

$$\Delta_{2m}^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}[\,y_0T - d_2 + \int_0^T Lg(t,\gamma(t,\alpha),$$

$$\beta(t,\alpha), y_m(t,x_0,y_0)\;\omega_m,\eta_m,v_m)\,dt]$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y_m(\tau,x_0,y_0),\omega_m,\,\eta_m,v_m)d\tau \;(38)$$

Theorem 3.1. Let all assumptions and conditions of the theorem 2.1 satisfied, then we get the following inequality :-

$$\begin{pmatrix} \|\,\Delta_1^{**}(0,x_0,y_0) - \Delta_{1m}^{**}(0,x_0,y_0)\,\| \\ \|\,\Delta_2^{**}(0,x_0,y_0) - \Delta_{2m}^{**}(0,x_0,y_0)\,\| \end{pmatrix} \le$$

$$\begin{pmatrix} \langle (S_1M_\gamma M_\beta F_1 \quad S_1M_\gamma M_\beta F_2), Q^m(E-Q)^{-1}\psi_1 \rangle \\ \langle (S_1N_\gamma N_\beta F_3 \quad S_1N_\gamma N_\beta F_4), Q^m(E-Q)^{-1}\psi_1 \rangle \end{pmatrix} \qquad (39)$$

where $S_1 = 1 + \frac{2}{T-2\tau}\alpha(t)$

Proof. By the equations (35) and (37), we get

$$\|\,\Delta_1^{**}(0,x_0,y_0) - \Delta_{1m}^{**}(0,x_0,y_0)\,\| = \Bigg\|\frac{2}{T(T-2\tau)}[\,x_0T$$

$$-d_1 + \int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha),\mu,\varepsilon,u)dt\,] + \frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,$$

$$\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),\int_a^b \gamma(\tau,\alpha)\,x(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)$$

$$x(\tau,x_0,y_0)d\tau,\int_{-\infty}^t R(t,\tau)(x(\tau,x_0,y_0)\,-y(\tau,x_0,y_0)d\tau\,)d\tau$$

$$-\frac{2}{T(T-2\tau)}[\,x_0T - d_1 + \int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha),$$

$$x_m(t,x_0,y_0),\mu_m,\varepsilon_m,u_m)dt\,]$$

$$-\frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x_m(\tau,x_0,y_0),\int_a^b \beta(\tau,\alpha)$$

$$x_m(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)\,x_m(\tau,x_0,y_0)d\tau,\int_{-\infty}^t R(t,\tau)$$

$$(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0))d\tau)d\tau\Bigg\|$$

$$\le \frac{2}{T(T-2\tau)}\int_0^T \Bigg[(1-\frac{t-\tau}{T})\int_{\tau}^t \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|$$

$$\Bigg[(K_1 + K_2M_\gamma(b-a) + K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)$$

$$\|x(\tau,x_0,y_0) - x_m(\tau,x_0,y_0)\|$$

$$+\frac{h}{\alpha}K_4\|y(\tau,x_0,y_0) - x_m(\tau,x_0,y_0)\|\Bigg]d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\,\|\beta(\tau,\alpha)\|\Bigg[(K_1 + K_2M_\gamma(b-a)$$

$$+K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)\|x(\tau,x_0,y_0) - x_m(\tau,x_0,y_0)\|$$

$$+\frac{h}{\alpha}K_4\|y(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\|\Bigg]d\tau\Bigg]dt$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} \|\gamma(\tau,\alpha)\|\,\|\beta(\tau,\alpha)\|\;\Bigg[(K_1 + K_2M_\gamma(b-a)$$

$$+K_3M_\beta(b-a) + \frac{h}{\alpha}K_4)\|x(\tau,x_0,y_0) - x_m(\tau,x_0,y_0)\|$$

$$+\frac{h}{\alpha}K_4\|y(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\|\Bigg]d\tau$$

hence

$$\|\,\Delta_1^{**}(0,x_0,y_0) - \Delta_{1m}^{**}(0,x_0,y_0)\,\| \le$$

$$\langle\,(S_1M_\gamma M_\beta F_1 \qquad S_1M_\gamma M_\beta F_2), Q^m(E-Q)^{-1}\psi_1\,\rangle \qquad (40)$$

similarly, we have

$$\|\,\Delta_2^{**}(0,x_0,y_0) - \Delta_{2m}^{**}(0,x_0,y_0)\,\|$$

$$\langle\,(S_1N_\gamma N_\beta F_3 \qquad S_1N_\gamma N_\beta F_4), Q^m(E-Q)^{-1}\psi_1\,\rangle \qquad (41)$$

i.e. the inequality (39) will be satisfied for all $m \ge 0$.

Theorem 3.2. Let the system (1) be defined on the interval $[a,b]$ and $[c,d]$ in R^1 and periodic in t of period T. Suppose that the sequence of functions (9) and (10) satisfies the following inequalities :-

$$\left.\begin{array}{l} \min\limits_{x_0\in J_1, y_0\in J_2} \Delta_{1m}^{**}(0,x_0,y_0) \le \\ -\langle (S_1 M_\gamma M_\beta F_1 \quad S_1 M_\gamma M_\beta F_2), Q^m (E-Q)^{-1}\psi_1\rangle \\ \max\limits_{x_0\in J_1, y_0\in J_2} \Delta_{1m}^{**}(0,x_0,y_0) \ge \\ \langle (S_1 M_\gamma M_\beta F_1 \quad S_1 M_\gamma M_\beta F_2), Q^m (E-Q)^{-1}\psi_1\rangle \end{array}\right\} \quad (42)$$

$$\left.\begin{array}{l} \min\limits_{x_0\in J_1, y_0\in J_2} \Delta_{2m}^{**}(0,x_0,y_0) \le \\ -\langle (S_1 N_\gamma N_\beta F_3 \quad S_1 N_\gamma N_\beta F_4), Q^m (E-Q)^{-1}\psi_1\rangle \\ \max\limits_{x_0\in J_1, y_0\in J_2} \Delta_{2m}^{**}(0,x_0,y_0) \ge \\ \langle (S_1 N_\gamma N_\beta F_3 \quad S_1 N_\gamma N_\beta F_4), Q^m (E-Q)^{-1}\psi_1\rangle \end{array}\right\} \quad (43)$$

where $x_0 \in J_1 = \left[a + \frac{T}{2} M_\gamma M_\beta M + \frac{2}{T-2\tau} q(x_0)\right.$

$\left., b - \frac{T}{2} M_\gamma M_\beta M - \frac{2}{T-2\tau} q(x_0)\right]$

and

$y_0 \in J_2 = \left[c + \frac{T}{2} N_\gamma N_\beta N + \frac{2}{T-2\tau} p(y_0)\right.$

$\left., d - \frac{T}{2} N_\gamma N_\beta N - \frac{2}{T-2\tau} p(y_0)\right]$

Then (1) has a periodic solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$

for $x_0 \in [a + J_1, b - J_1]$ and $y_0 \in [c + J_2, d - J_2]$.

Proof . Let x_1, x_2 and y_1, y_2 be any points belonging on the intervals I_1 and I_2 respectively, such that

$$\left.\begin{array}{l} \Delta_{1m}^{**}(0,x_1,y_1) = \min\limits_{x_0\in I_1, y_0\in I_2} \Delta_{1m}^{**}(0,x_0,y_0) \\ \Delta_{1m}^{*}(0,x_2,y_2) = \max\limits_{x_0\in I_1, y_0\in I_2} \Delta_{1m}^{**}(0,x_0,y_0) \end{array}\right\} \quad (44)$$

$$\left.\begin{array}{l} \Delta_{2m}^{**}(0,x_1,y_1) = \min\limits_{x_0\in I_1, y_0\in I_2} \Delta_{2m}^{**}(0,x_0,y_0) \\ \Delta_{2m}^{**}(0,x_2,y_2) = \max\limits_{x_0\in I_1, y_0\in I_2} \Delta_{2m}^{**}(0,x_0,y_0) \end{array}\right\} \quad (45)$$

By using the inequalities (40), (41), (42), and (43), we obtains

$$\left.\begin{array}{l} \Delta_1^{**}(0,x_1,y_1) = \Delta_{1m}^{**}(0,x_1,y_1) + \\ \quad (\Delta_1^{**}(0,x_1,y_1) - \Delta_{1m}^{**}(0,x_1,y_1)) < 0 \\ \Delta_1^{**}(0,x_1,y_1) = \Delta_{1m}^{**}(0,x_2,y_2) + \\ \quad (\Delta_1^{**}(0,x_2,y_2) - \Delta_{1m}^{**}(0,x_2,y_2)) > 0 \end{array}\right\} \quad (46)$$

$$\left.\begin{array}{l} \Delta_2^{**}(0,x_1,y_1) = \Delta_{2m}^{**}(0,x_1,y_1) + \\ \quad + (\Delta_2^{**}(0,x_1,y_1) - \Delta_{2m}^{**}(0,x_1,y_1)) < 0 \\ \Delta_2^{**}(0,x_1,y_1) = \Delta_{2m}^{**}(0,x_2,y_2) + \\ \quad + (\Delta_2^{**}(0,x_2,y_2) - \Delta_{2m}^{**}(0,x_2,y_2)) > 0 \end{array}\right\} \quad (47)$$

and from the continuity of the functions $\Delta_1^{**}(0,x_0,y_0)$ and $\Delta_2^{**}(0,x_0,y_0)$ and the inequalities (3.10) and (3.11), then there exist and isolated singular points $x^0 \in [x_1, x_2]$ and $y^0 \in [y_1, y_2]$ such that $\Delta_1^{**}(0,x_0,y_0) = \Delta_2^{**}(0,x_0,y_0) = 0$. This means that (1) has a periodic solution $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$.

4. STABILITY PERIODIC SOLUTION OF THE SYSTEM (1)

Theorem 4.1. If the function $\Delta_1^{**}(0,x_0,y_0)$ and $\Delta_2^{**}(0,x_0,y_0)$ are defined by :-

$\Delta_1^{**}(0,x_0,y_0) : D_{1I} \times D_{2I} \to R^n$

$$\Delta_1^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}\Big[x_0 T - d_1 + \int_0^T Lf(t,\gamma(t,\alpha), \beta(t,\alpha), x(t,x_0,y_0), \mu, \varepsilon, u) dt\Big] + \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),\mu,\varepsilon,u)d\tau \quad (48)$$

and

$\Delta_2^{**}(0,x_0,y_0) : D_{1I} \times D_{2I} \to R^n$

$$\Delta_2^{**}(0,x_0,y_0) = \frac{2}{T(T-2\tau)}\Big[y_0 T - d_2 + \int_0^T Lg(t,\gamma(t,\alpha), \beta(t,\alpha), y(t,x_0,y_0), \omega, \eta, v) dt\Big] + \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0,y_0),\omega,\eta,v)d\tau \quad (49)$$

where $\begin{pmatrix} x(t,x_0,y_0) \\ y(t,x_0,y_0) \end{pmatrix}$ is the limits of the sequences (9) and (10) respectively, the following inequality holds :-

$$\begin{pmatrix} \|\Delta_1^{**}(0,x_0,y_0)\| \\ \|\Delta_2^{**}(0,x_0,y_0)\| \end{pmatrix} \le \begin{pmatrix} M_\gamma M_\beta M + \frac{2}{T(T-2\tau)} q(x_0) \\ N_\gamma N_\beta N + \frac{2}{T(T-2\tau)} p(y_0) \end{pmatrix} \quad (50)$$

$$\|\Delta_1^{**}(0,x_0^1,y_0^1) - \Delta_1^{**}(0,x_0^2,y_0^2)\| \le I_1 \|x_0^1 - x_0^2\| + I_2 \|y_0^1 - y_0^2\| \quad (51)$$

and

$$\|\Delta_2^{**}(0,x_0^1,y_0^1) - \Delta_2^{**}(0,x_0^2,y_0^2)\| \le I_3 \|x_0^1 - x_0^2\| + I_4 \|y_0^1 - y_0^2\| \quad (52)$$

$E_1 = \frac{T}{2} M_\gamma M_\beta F_2 r_1^2 u_1 u_2$, $E_2 = \frac{T}{2} N_\gamma N_\beta F_3 r_1^2 u_1 u_2$,

$I_1 = S_2 + M_\gamma M_\beta S_1 u_3 (F_1 r_1 u_1 + F_2 E_2)$

$, I_2 = M_\gamma M_\beta S_1 u_3 (F_1 E_1 + F_2 r_1 u_2)$,

$I_3 = N_\gamma N_\beta S_1 u_3 (F_3 r_1 u_1 + F_4 E_2)$,

$I_4 = S_2 + N_\gamma N_\beta S_1 u_3 (F_3 E_1 + F_4 r_1 u_2)$,

$u_1 = \left(1 - \frac{T}{2} M_\gamma M_\beta F_1 r_1\right)^{-1}, u_2 = \left(1 - \frac{T}{2} N_\gamma N_\beta F_4 r_1\right)^{-1}$,

$u_3 = (1 - \frac{T^2}{4} M_\gamma N_\gamma M_\beta N_\beta F_2 F_3 r_1^2 u_1 u_2)^{-1}, , S_2 = \frac{2}{T-2\tau}$

Proof. From (48), we have

$$\|\Delta_1^{**}(0,x_0,y_0)\| = \left\|\frac{2}{T(T-2\tau)}\Big[x_0 T - d_1 + \int_0^T Lf(t,\gamma(t,\alpha),\right.$$

$$\beta(t,\alpha)\,, x(t,x_0,y_0),\mu,\varepsilon,u)d\tau\,]$$

$$+\frac{1}{T}\int_{\tau}^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0,y_0),$$

$$\int_a^b \gamma(\tau,\alpha)x(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)x(\tau,x_0,y_0)\,d\tau,$$

$$\int_{-\infty}^{t} R(t,\tau)\left(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\right)d\tau)d\tau\Bigg\|$$

$$\leq \frac{2}{T(T-2\tau)}[\,\|x_0T-d_1\|+\int_0^T\Bigg[\left(1-\frac{t-\tau}{T}\right)\int_\tau^t \|\gamma(\tau,\alpha)\|$$

$$\|\beta(\tau,\alpha)\|\Bigg\|f(\tau,\ x(\tau,x_0,y_0),\int_a^b \gamma(\tau,\alpha)\,x(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)$$

$$x(\tau,x_0,y_0)d\tau,\int_{-\infty}^{t} R(t,\tau)\left(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\right)d\tau\Bigg\|d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T}\|\gamma(\tau,\alpha)\|\,\|\beta(\tau,\alpha)\|\Bigg\|f(\tau,x(\tau,x_0,y_0),\int_a^b \gamma(\tau,\alpha)$$

$$x(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)x(\tau,x_0,y_0)d\tau,\ \int_{-\infty}^{t} R(t,\tau)$$

$$\left(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\right)d\tau)d\tau\Bigg\|\,d\tau\Bigg]dt\,]$$

$$+\frac{1}{T}\int_\tau^{\tau+T}\|\gamma(\tau,\alpha)\|\,\|\beta(\tau,\alpha)\|\Bigg\|f(\tau,x(\tau,x_0,y_0),\int_a^b \gamma(\tau,\alpha)$$

$$x(\tau,x_0,y_0)d\tau,\int_a^b \beta(\tau,\alpha)\,x(\tau,x_0,y_0)d\tau,\int_{-\infty}^{t} R(t,\tau)$$

$$\left(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\right)d\tau)d\tau\Bigg\|\,d\tau$$

so, we have

$$\|\,\Delta_1^{**}(0,x_0,y_0)\,\|\leq\ M_\gamma M_\beta M+\frac{2}{T(T-2\tau)}q(x_0) \qquad (53)$$

Similarly, by using (49), we get

$$\|\,\Delta_2^{**}(0,x_0,y_0)\,\|\leq\ N_\gamma N_\beta N+\frac{2}{T(T-2\tau)}p(y_0) \qquad (54)$$

Also rewrite (53) and (54) in a vector form we get (50).

Again from (48), we obtain

$$\|\,\Delta_1^{**}(0,x_0^1,y_0^1)-\Delta_1^{**}(0,x_0^2,y_0^2)\,\|=\Bigg\|\frac{2}{T(T-2\tau)}[\,x_0^1T-d_1$$

$$+\int_0^T Lf(t,\gamma(t,\alpha)\,,\beta(t,\alpha),x(t,x_0^1,y_0^1),\mu,\varepsilon,u)dt\,]$$

$$+\frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0^1,y_0^1),\int_a^b \gamma(\tau,\alpha)$$

$$x(\tau,x_0^1,y_0^1)d\tau,\int_a^b \beta(\tau,\alpha)\ x(\tau,x_0^1,y_0^1)d\tau,\int_{-\infty}^{t} R(t,\tau)$$

$$\left(x(\tau,x_0^1,y_0^1)-y(\tau,x_0^1,y_0^1)\right)d\tau)d\tau$$

$$-\frac{2}{T(T-2\tau)}[\,x_0^2T-d_1+\int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha),$$

$$x(t,x_0^2,y_0^2),\mu,\varepsilon,u)dt\,]$$

$$-\frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha)\,,x(\tau,x_0^2,y_0^2),\int_a^b \gamma(\tau,\alpha)$$

$$x(\tau,x_0^2,y_0^2)d\tau,\int_a^b \beta(\tau,\alpha)\ x(\tau,x_0^2,y_0^2)d\tau,\int_{-\infty}^{t} R(t,\tau)$$

$$\left(x(\tau,x_0^2,y_0^2)-y(\tau,x_0^2,y_0^2)\right)d\tau)\Bigg\|$$

$$\leq\ \frac{2}{T-2\tau}\|x_0^1-x_0^2\|+\frac{2}{T(T-2\tau)}\int_0^T\Bigg[(1-\frac{t-\tau}{T})$$

$$\int_\tau^t\|\gamma(\tau,\alpha)\|\,\|\beta(\tau,\alpha)\|\Bigg[(K_1+K_2M_\gamma(b-a)+K_3M_\beta(b-a)$$

$$+\frac{h}{\alpha}K_4)\|x(\tau,x_0^1,y_0^1)-x(\tau,x_0^2,y_0^2)\|+\frac{h}{\alpha}K_4$$

$$\|y(\tau,x_0^1,y_0^1)-y(\tau,x_0^2,y_0^2)\|\Bigg]d\tau$$

$$+\frac{t-\tau}{T}\int_t^{\tau+T}\|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|\Bigg[(K_1+K_2M_\gamma(b-a)$$

$$+K_3M_\beta(b-a)+\frac{h}{\alpha}K_4)\|x(\tau,x_0^1,y_0^1)-x(\tau,x_0^2,y_0^2)\|$$

$$+\frac{h}{\alpha}K_4\|y(\tau,x_0^1,y_0^1)-y(\tau,x_0^2,y_0^2)\|\Bigg]d\tau\Bigg]dt$$

$$+\frac{1}{T}\int_\tau^{\tau+T}\|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\|\Bigg[(K_1+K_2M_\gamma(b-a)$$

$$+K_3M_\beta(b-a)+\frac{h}{\alpha}K_4)\ \|x(\tau,x_0^1,y_0^1)-x(\tau,x_0^2,y_0^2)\|$$

$$+\frac{h}{\alpha}K_4\|y(\tau,x_0^1,y_0^1)-y(\tau,x_0^2,y_0^2)\|\Bigg]d\tau$$

So, the last inequality becomes

$$\|\,\Delta_1^{**}(0,x_0^1,y_0^1)-\Delta_1^{**}(0,x_0^2,y_0^2)\,\|\leq\ S_2\|x_0^1-x_0^2\|$$

$+M_\gamma M_\beta S_1 F_1 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + M_\gamma M_\beta S_1 F_2$

$\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\|$ (55)

Also, by using equation (4.2), we have

$\| \Delta_2^{**}(0,x_0^1,y_0^1) - \Delta_2^{**}(0,x_0^2,y_0^2) \| \le S_2 \|y_0^1 - y_0^2\|$

$+N_\gamma N_\beta S_1 F_3 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + N_\gamma N_\beta S_1 F_4$

$\|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\|$ (56)

where $x(t,x_0^1,y_0^1), x(t,x_0^2,y_0^2), y(t,x_0^1,y_0^1)$ and $y(t,x_0^2,y_0^2)$ are the solutions of the integral equations :-

$$x(t,x_0^k,y_0^k) = x_0^k + \int_\tau^t \Big[f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0^k,y_0^k),\mu,\varepsilon, u) + \frac{1}{T}\int_\tau^{\tau+T} f(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),x(\tau,x_0^k,y_0^k),\mu,\varepsilon,u)d\tau \Big] d\tau$$
$$+ \frac{2(t-\tau)}{T(T-2\tau)} \Big[d_1 - x_0^k T - \int_0^T Lf(t,\gamma(t,\alpha),\beta(t,\alpha), x(t,x_0^k,y_0^k),\mu,\varepsilon,u)dt \Big] \quad (57)$$

and

$$y(t,x_0^k,y_0^k) = y_0^k + \int_\tau^t \Big[g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0^k,y_0^k,\omega,\eta, v) - \frac{1}{T}\int_\tau^{\tau+T} g(\tau,\gamma(\tau,\alpha),\beta(\tau,\alpha),y(\tau,x_0^k,y_0^k),\omega,\eta,v)d\tau \Big] d\tau$$
$$+ \frac{2(t-\tau)}{T(T-2\tau)} \Big[d_2 - y_0^k T - \int_0^T Lg(t,\gamma(t,\alpha),\beta(t,\alpha), y(t,x_0^k,y_0^k),\omega,\eta,v)dt \Big] \quad (58)$$

with

$x_0^k(t,x_0^k,y_0^k) = x_0^k, y_0^k(t,x_0^k,y_0^k) = y_0^k$ where $k = 1,2$.

From the equation (58) and using Lemma 1.1, we get

$$\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \le \|x_0^1 - x_0^2\| + (1 - \frac{t-\tau}{T}) \int_\tau^t \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Big[(K_1 + K_2 M_\gamma(b-a) + K_3 M_\beta(b-a) + \frac{h}{\alpha}K_4) \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| + \frac{h}{\alpha}K_4 \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\| \Big] d\tau$$
$$+ \frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Big[(K_1 + K_2 M_\gamma(b-a) + K_3 M_\beta(b-a) + \frac{h}{\alpha}K_4) \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| + \frac{h}{\alpha}K_4 \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\| \Big] d\tau$$
$$+ \frac{2}{T-2\tau}\int_0^T \Big[(1 - \frac{t-\tau}{T}) \int_\tau^t \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Big[(K_1 + K_2 M_\gamma(b-a) + K_3 M_\beta(b-a) + \frac{h}{\alpha}K_4) \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| + \frac{h}{\alpha}K_4 \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\| \Big] d\tau$$
$$+ \frac{t-\tau}{T}\int_t^{\tau+T} \|\gamma(\tau,\alpha)\|\|\beta(\tau,\alpha)\| \Big[(K_1 + K_2 M_\gamma(b-a) + K_3 M_\beta(b-a) + \frac{h}{\alpha}K_4 \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| + \frac{h}{\alpha}K_4 \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\| \Big] d\tau \Big] dt$$
$$\le (1 - \frac{T}{2}M_\gamma M_\beta F_1 r_1)^{-1} r_1 \|x_0^1 - x_0^2\| + \frac{T}{2}M_\gamma M_\beta F_2 r_1 (1 - \frac{T}{2}M_\gamma M_\beta F_1 r_1)^{-1} \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\|$$

so that

$$\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \le r_1 u_1 \|x_0^1 - x_0^2\| + \frac{T}{2}M_\gamma M_\beta F_2 r_1 u_1 \|y(\tau,x_0^1,y_0^1) - y(\tau,x_0^2,y_0^2)\| \quad (59)$$

Now, from (58), we get

$$\|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \le r_1 u_2 \|y_0^1 - y_0^2\| + \frac{T}{2}N_\gamma N_\beta F_3 r_1 u_2 \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| \quad (60)$$

Substituting the last inequality in (59), as follows

$$\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \le r_1 u_1 \|x_0^1 - x_0^2\| + \frac{T}{2}M_\gamma M_\beta F_2 r_1 u_1 \, [\, r_1 u_2 \|y_0^1 - y_0^2\| + \frac{T}{2}N_\gamma N_\beta F_3 r_1 u_2 \|x(\tau,x_0^1,y_0^1) - x(\tau,x_0^2,y_0^2)\| \,]$$
$$\le (1 - \frac{T^2}{4}M_\gamma N_\gamma M_\beta N_\beta F_2 F_3 r_1^2 u_1 u_2)^{-1} r_1 u_1 \|x_0^1 - x_0^2\| + \frac{T}{2}M_\gamma M_\beta F_2 r_1^2 u_1 u_2 \, (1 - \frac{T^2}{4}M_\gamma N_\gamma M_\beta N_\beta F_2 F_3 r_1^2 u_1 u_2)^{-1} \|y_0^1 - y_0^2\|$$

Then we write this equation as follows

$$\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\|\le r_1u_1u_3\|x_0^1-x_0^2\| + E_1u_3\|y_0^1-y_0^2\| \quad (61)$$

Also, substituting (60) in (61), we obtain

$$\|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\|\le E_2u_3\|x_0^1-x_0^2\| + r_1u_2u_3\|y_0^1-y_0^2\| \quad (62)$$

So, substituting the inequalities (61) and (62) in inequality (55), we get the inequality (51).

Similarly, substituting the inequalities (61) and (62) in inequality (56), we get the inequality (52).

Theorem 4.2. Let the system (1) be defined in the domain (3). Suppose that G_1 and G_2 be closed and bounded domain subset of domain D_{1I} and D_{2I}. Then, G_1 and G_2 have points at which the Δ −constant is zero, then for any point $x_0 \in D_{1I}$ and $y_0 \in D_{2I}$, the following inequality holds :-

$$\|\Delta_1^{**}(0,x_0,y_0)\| \le \langle (S_1M_\gamma M_\beta F_1 \quad S_1M_\gamma M_\beta F_2), Q^m (E-Q)^{-1}\psi_1 \rangle + M_\gamma M_\beta M + \frac{2}{T(T-2\tau)}q(x_0) \quad (63)$$

and

$$\|\Delta_{2m}^{**}(0,x_0,y_0)\| \le \langle (S_1N_\gamma N_\beta F_3 \quad S_1N_\gamma N_\beta F_4), Q^m (E-Q)^{-1}\psi_1 \rangle + N_\gamma N_\beta N + \frac{2}{T(T-2\tau)}p(y_0) \quad (64)$$

for all $m \ge 0$ and $x_0 \in D_{3I}, y_0 \in D_{4I}$.

Proof . By using the inequality (39), we get

$$\|\Delta_{1m}^{**}(0,x_0,y_0)\| = \|\Delta_{1m}^{**}(0,x_0,y_0) - \Delta_1^{**}(0,x_0,y_0) + \Delta_1^{**}(0,x_0,y_0)\|$$

$$\le \|\Delta_{1m}^{**}(0,x_0,y_0) - \Delta_1^{**}(0,x_0,y_0)\| + \|\Delta_1^{**}(0,x_0,y_0)\|$$

$$\le \langle (S_1M_\gamma M_\beta F_1 \quad S_1M_\gamma M_\beta F_2), Q^m(E-Q)^{-1}\psi_1 \rangle + M_\gamma M_\beta M + \frac{2}{T(T-2\tau)}q(x_0)$$

Again from (39), we get

$$\|\Delta_{2m}^{**}(0,x_0,y_0)\| = \|\Delta_{2m}^{**}(0,x_0,y_0) - \Delta_2^{**}(0,x_0,y_0) + \Delta_2^{**}(0,x_0,y_0)\|$$

$$\le \|\Delta_{2m}^{**}(0,x_0,y_0) - \Delta_2^{**}(0,x_0,y_0)\| + \|\Delta_2^{**}(0,x_0,y_0)\|$$

$$\le \langle (S_1N_\gamma N_\beta F_3 \quad S_1N_\gamma N_\beta F_4), Q^m(E-Q)^{-1}\psi_1 \rangle + N_\gamma N_\beta N + \frac{2}{T(T-2\tau)}p(y_0)$$

5. CONCLUSION

In this paper, we have established the existence and approximation of the periodic solutions for non-linear integro-differential equations depended on special functions with singular kernels and boundary integral conditions. The numerical-analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by (Samoilenko, A. M.). Also, we expand the result obtained by Samoilenko to change the periodic system of non-linear differential equations with intial condition to periodic system of non-linear differential equations with boundary integral conditions.

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