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Improved Numerical Computation to Solve Lane-Emden type Equations

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ABSTRACT

Lane-Emden equation is also of fundamental importance in mathematical physics, celestial mechanics, and computer science. It can be used to describe stellar structures, equilibrium density distribution in polytrophic isothermal gas, thermal behavior in mutual attraction of its molecules. An improved numerical method is developed for solving Lane-Emden type differential equations. The method is based on power series solutions of differential equations and Maclaurin series expansion. A python program is written to carry out numerical calculations. Five examples are solved and shown in this paper, the solutions obtained by the program are compared with the exact solutions of differential equation, an excellent agreement is found between them. The present method improves runtime.

Key words: Lane-Emden equation; Power series solution; Astrophysics; numerical solution of differential equation; nonlinear differential equation; Maclaurin Series; Stellar Structures, Polytropic isothermal gas

1. INTRODUCTION

In physics we mostly come across second order differential equation, which represent a particular system and its behavior. A generalized Lane-Emden equation is given by

$$y'' + \frac{m}{x} y' + h(x) f(y) = g(x), \quad 0 \le x \le 1,$$

$$y(0) = \alpha, y'(0) = \beta$$

It is non-homogenous second order differential equation. Both the functions $f(y), h(x)$ and $g(x)$ are continuous and differentiable real valued functions. Lane Emden equation is homogenous in some cases where the function $g(x)$ is zero. α and β are initial values of y and its derivatives at x = 0.Lane Emden equation is nonlinear differential equation, which has regular singularity at x = 0. Lane–Emden type equation models many phenomena in physics, chemistry, and astronomy. It plays a fundamental role in explaining the structures of stars and planets[1]. It can be used to model equilibrium density distribution of polytrophic isothermal gas and diffusion process[2-3]. The singularity at origin makes it of fundamental importance modeling of cluster of galaxies. Mohan & Bayaty presented a power series solution of the Lane-Emden equation based on Tylor series [4]. Pascual presented solution of some of the LE equation using Pade approximation [5]. Horedt solved LE equation for polytrope using Bessel functions, he is also discussed the solution for imaginary surfaces [6]. In last decade scientists used Hermite, Legendre and Laguerre polynomials, Hermite spline functions, collocation methods, and differential transform methods to solve LE differential type equation [7-18].

2. METHODOLOGY /ALGORITHM

The numerical solution is obtained by power series solution technique. A computer program in python is developed to solve Lane-Emden type equation (1) for m = 2. Following are steps through which solution is obtained, Step 1: The functions $f(x,y)$ and $g(x)$ are analytical functions. These functions first are expended in power series by Maclaurin series expansion.

Step 2: The solution y and its derivatives $y''and\ y'$ are expended in powers of ‘x’ using series given below,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Step 3: The values of steps 1 and 2 are plugged in LE equation.

Step 4: The coefficients a_n are found by comparing coefficients of powers of ‘x’.

3. APPLICATION

Lane an U.S. Astrophysicist first introduced this equation in astrophysics to calculated mass density and temperature of a surface [19]. Later in 1907 Emden studied this equation [20]. LE equation is considered as dimensionless version of Poisson equation for gravitational potential of a stellar surfaces. LE equation is also of fundamental importance in mathematical physics, celestial mechanics. It can be used to describe stellar structures, equilibrium density distribution in polytropic isothermal gas, thermal behavior in mutual attraction of its molecules.

A spherical gas cloud of radius ‘r’ has hydraulic pressure dues to usual pressure of gas and pressure contribution due to photon radiations. The change in pressure is related to change is potential ‘φ’ and is given by $P = \rho d\varphi$, where φ satisfies Poisson equation,

$$\nabla^2 \varphi = 4\pi G \rho$$

By simple substitutions, the above equation can be written as

$$y'' + \frac{2}{x} y' + y^m(x) = 0, \quad \#(1)$$

Where $\rho = \rho_0 y^m$, ρ_0 is the density of the cloud at the center. With initial condition $y(0) = 1\ and\ y'(0) = 0$, equation is called Lane-Emden equation. For $m = 1$, equation (1) becomes,

The method given above is used to solve five differential equations given in examples 1-5. In example 1, LE equation given in equation (1) is solved for $m = 1$.

Example 1:$y'' + \frac{2}{x} y' + y(x) = 0, \quad 0 \le x \le 1,$

$$y(0) = 1, y'(0) = 0$$

Here $h(x) = 1,\ g(x) = 0$

The program generates a solution given below,

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots$$

The exact solution $sinx/x$ [21]. The difference between two values obtained by exact and computer-generated solution at x =1 is 2.49 x 10^{-8}. The fig. 1 shows an excellent agreement between graph of exact and solution obtained by computer program. The runtime is 0.0039 second.

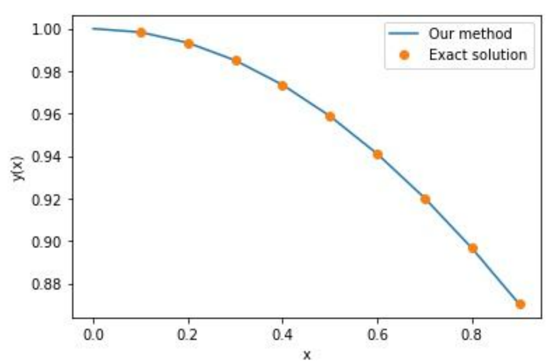

Fig. 1: Plot of numerical and exact solutions of example 1.

Example 2:$y'' + \frac{2}{x} y' = 2(2x^2 + 3)y(x), \quad 0 \le x \le 1,$

$$y(0) = 1, y'(0) = 0$$

Here $h(x) = -2(2x^2 + 3)$, $g(x) = 0$

The solution obtained by our method is

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \frac{x^{12}}{6!} + \cdots$$

Whereas the exact solution is $\exp(x^2)$. Fig. 2 shows plots of exact and generated solution, the solution is in good agreement. Table 1 shows a comparison of runtimes of our work and ref. [8] for various 'n'. The maximum error is same in two methods but runtimes for our program is much less than that of ref. [8]

Table 1: Comparison of runtime for calculation by our method and method of ref. [8].

n	Maximum error	Runtime	
		Ref. [8]	This work
12	2.26×10^{-4}	0.234	0.0169
16	3.05×10^{-6}	0.358	0.0219
20	2.73×10^{-8}	0.436	0.0369
24	1.72×10^{-10}	0.593	0.0408
28	8.15×10^{-13}	0.702	0.0458
32	2.00×10^{-15}	0.974	0.0528

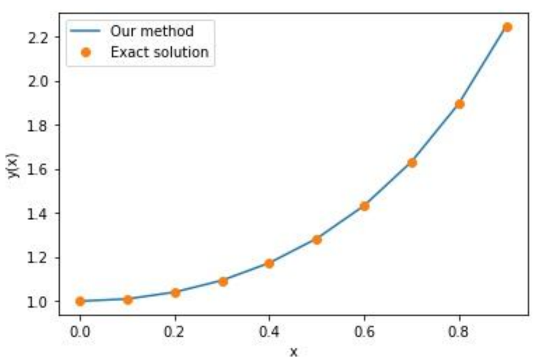

Fig. 2: Plot of numerical and exact solutions of example 2.

Example 3: $y'' + \frac{2}{x}y' + y(x) = (2x^2 + 6x + 6)e^x, \quad 0 \le x \le 1,$

$$y(0) = 0, y'(0) = 0$$

Here $h(x) = 1,\ g(x) = (2x^2 + 6x + 6)e^x$

The program generates a solution given below,

$$x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \frac{x^6}{24} + \frac{x^7}{120} + \cdots$$

Which is same as the exact solution x^2e^x. The fig. 3 gives graph of exact and solution obtained by computer program, it shows an excellent agreement between them. The runtime is 0.01 second.

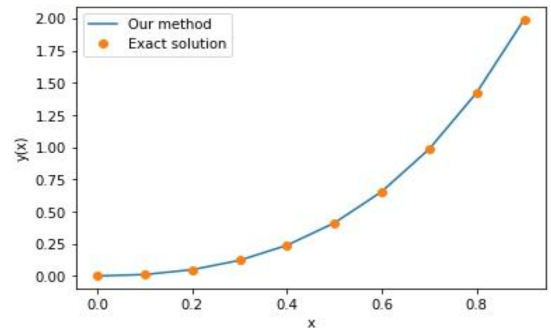

Fig. 3: Plot of numerical and exact solutions of example 3.

Example 4: $y'' + \frac{2}{x}y' + y(x) = 6(\cos(x) - x\sin(x)), \quad 0 \le x \le 1,$

$$y(0) = 0, y'(0) = 0$$

Here $h(x) = 1,\ g(x) = 6(\cos(x) - x\sin(x))$

The program generates a solution given below,

$$x^2 + \frac{x^4}{2!} + \frac{x^6}{4!} + \frac{x^8}{6!} + \cdots$$

The exact solution is $x^2 cosx$. The difference between two values obtained by exact and computer-generated solution at x =1 is 4.7×10^{-14}. The fig. 4 shows an excellent agreement between graph of exact and solution obtained by computer program. The runtime is 0.093 second.

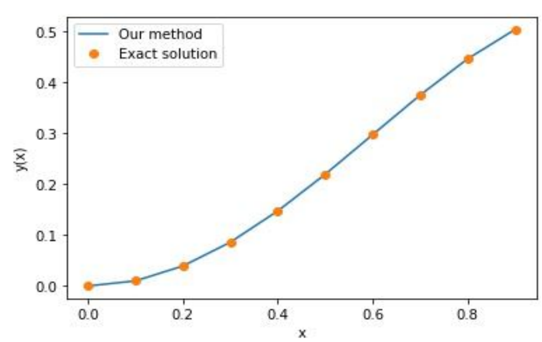

Fig. 4: Plot of numerical and exact solutions of example 4.

Example 5: $y'' + \frac{2}{x}y' + y(x) = 6 + 12x + x^2 + x^3$, $0 \le x \le 1$,

$$y(0) = 0, y'(0) = 0$$

Here $h(x) = 1$, $g(x) = 6 + 12x + x^2 + x^3$

Expanding $y(x)$ into few terms of power series in 'x' the computer program gives exact solution

$$y(x) = x^2 + x^3$$

The series was expanded up to five terms, the runtime is 0.0059 second, which shows the efficiency of our program, the runtime for polynomial method [8], homology perturbation method [22], and Adomian decomposition method [23] are 0.015, 0.067, and 0.078 seconds, respectively.

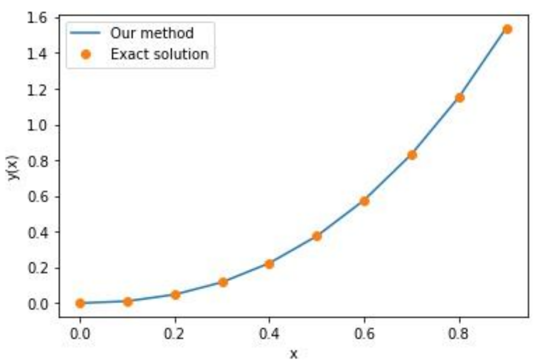

Fig. 5: Plot of numerical and exact solutions of example 5.

4. CONCLUSION

In this paper an improved numerical method to solve Lane Emden type equation is discussed. Five Lane Emden type differential equations are solved by the computer program developed in python based on the methodology given above. The method is based on power series solutions. The functions h(x) and g(x) are expanded in power series using Maclurins series expansion. The advantage of this method is that it improves runtime. The solution obtained by python program is compared with the exact solution of LE differential equations, an excellent agreement between them is seen.

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