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Computer Simulation and Visualization of the Dynamic Poroelasticity Problem Solutions

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ABSTRACT

In this paper, we numerically computed the analytical solution of the initial-boundary poroelasticity problem. We applied parallel computation for inverse Laplace transformation and consecutive computation forinverse Fourier transformation to obtainthe two-dimensional time-dependent solution. Visualization of the dynamic solution functions is presented for the fixed time values and confirms that the boundary conditions are satisfied at zero and in infinity.

Key words: Porous media, Fourier-Laplace transformations, poroelasticity theory, acoustic waves.

1. INTRODUCTION

Computer and mathematical modelling of a non-stationary physical processes taking place in porous mediums saturated with liquid plays an important role in studying the fluid flows in porous structures.

It is particularly important due to the complexity, both forexperimental and theoretical study, of the internal structure of porous medium.While a wide application of computer simulations based on realistic mathematical models drives further research in that direction [1]-[11]. The latest advances in such mathematical modelling and simulations help to develop many other areas of research, including earth and material sciences, mechanics, biotechnology and medicine,the theory of energy and filtration theory.Frenkel-Biottype theories [12], [13] areoften used for studying dynamic processes in porous media. Although, there is no sufficient evidence of comparison of the theoretical results obtained based on Frenkel-Biot theory with experimental results based on natural samples. An alternative continuous filtration theory based on methods of conservation laws and first physical principles was proposed by V.N.Dorovsky in 1989 [14], whereasFrenkel-Biot theory was built within a variation approach. Both theories consider propagation of three types of acoustic waves, two longitudinal and one transversal. In the Frenkel-Biot model, the velocities of seismic waves propagatingporous media are described by four elastic parameters for given physical parameters of the media. Whereasthe Dorovsky model obtained by linearization of the continuous filtration theory equations describes the porous medium saturated with liquid by only three elastic modules [15]-[16].In this paper we provide computer simulation of the solution functionsof a two-dimensional dynamic problem presented in the form of the hyperbolic inhomogeneous system of partial differential equations(PDE) with initial and boundary conditions.By application of the spectral method with analytical integral Fourier-Laplace transforms the solution of the initial PDE problem is reduced to solving the simpler ordinary differential equations problem, which has a derivative with respect to only one spatial variable. The sufficient condition of resolvability of the problem and the exact solution functions in explicit form are obtained in (17) by applying inverse Fourier-Laplace transforms. In [18], [19] another method is proposed for periodic solutions for nonlinear systems of integro-differential equations.

2. STATEMENT OF THE PROBLEM

Let us turn to the mathematical formulation of the model for the two-dimensional dynamic problem. The governing equations are based on the conservation and Hook's laws and are consistent with the thermodynamics conditions. We consider the half-plane $x_2 > 0$ filled with porous media and saturating liquid with parameters characterizing each of them. So, the propagation of seismic waves in these environments in the absence of loss of energy is described by the following initial-boundary problems in terms of the velocity of the saturating liquid, the velocities of the solid matrix, the liquid pressure, and the stress tensor.

The considered problem is stated as follows:

Momentum Conservation Law for an elastic medium:

$$\frac{\partial u_j}{\partial t} + \frac{1}{\rho_{0,s}}\left(\frac{\partial \sigma_{j1}}{\partial x_1} + \frac{\partial \sigma_{j2}}{\partial x_2}\right) + \frac{1}{\rho_0}\frac{\partial p}{\partial x_j} = F_j, (1)$$

Momentum Conservation Law for a liquid:

$$\frac{\partial v_j}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_j} = F_j, (2)$$

Hooke's Law for a solid matrix (elastic medium):

$$\frac{\partial \sigma_{jk}}{\partial t} + \mu\left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right) + \left(\frac{\rho_{0l}}{\rho_0}K - \frac{2}{3}\mu\right)\delta_{jk} div\vec{u} - \\ - \frac{\rho_{0,s}}{\rho_0} K \delta_{jk} div\vec{v} = 0, \quad (3)$$

Hooke's Law for a liquid :

$$\frac{\partial p}{\partial t} - (K - \alpha\rho_0\rho_{0,s}) div\vec{u} + \alpha\rho_0\rho_{0,l} div\vec{v} = 0, \quad (4)$$

Initial conditions:

$$u_j\big|_{t=0} = v_j\big|_{t=0} = \sigma_{jk}\big|_{t=0} = p\big|_{t=o} = 0, \quad (5)$$

Boundary conditions on a free surface in the plane $x_2 = 0$:

$$\sigma_{22} + p\big|_{x_2=0} = \sigma_{12}\big|_{x_2=0} = \frac{\rho_{0,l}}{\rho_0} p\Big|_{x_2=0} = 0, \quad (6)$$

For the two-dimensional case $j = 1,2;\ k = 1,2$ and the system (1)-(4) consists of eight partial differential equations with eight initial conditions (5) at $t = 0$ and three boundary conditions (6) at $x_2 = 0$, where $\vec{u} = (u_1, u_2)^T$ and $\vec{v} = (v_1, v_2)^T$ are the velocity vectors of an elastic porous medium and a liquid with the following physical parameters:

$\rho_{0,s}$ - partial density of an elastic porous medium;

$\rho_{0,l}$ -partial density of a saturating liquid,

$\rho_0 = \rho_{0,s} + \rho_{0,l}$,

$\rho_{0,s} = \rho_{0,s}^f (1 - d_0)$,

$\rho_{0,l} = \rho_{0,l}^f d_0$, where $\rho_{0,s}^f$ and $\rho_{0,l}^f$ are, respectively, the physical densities of the porous medium and of the liquid,

d_0 is the porosity

and p is the porous pressure,

σ_{jk} are the stress tensor components,

δ_{jk} is the Kronecker symbol.

Moreover, $K = \lambda + \frac{2}{3}\mu$, where $\lambda > 0$ and $\mu > 0$ are the Lame coefficients, $\alpha = \rho_0\alpha_3 + \frac{K}{\rho_0^2}$, $\rho_0^3\alpha_3 > 0$ is the bulk compression modulus of the liquid component of a heterophase medium.

Elastic modules $K > 0$, $\mu > 0$ and $\alpha_3 > 0$ $(K, \mu, \alpha \in \square)$ are expressed in terms of transverse wave propagation velocity c_s and two longitudinal waves velocities c_{p1}, c_{p2} as follows:

$$\mu = \rho_{0,s} c_s^2,$$

$$K = \frac{\rho_0}{2}\frac{\rho_{0,s}}{\rho_{0,l}}\left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3}\frac{\rho_{0,l}}{\rho_0}c_s^2 - \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9}\frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2}c_s^4}\right),$$

$$\alpha_3 = \frac{1}{2\rho_0^2}\left(c_{p_1}^2 + c_{p_2}^2 - \frac{8}{3}\frac{\rho_{0,s}}{\rho_0}c_s^2 + \sqrt{(c_{p_1}^2 - c_{p_2}^2)^2 - \frac{64}{9}\frac{\rho_{0,l}\rho_{0,s}}{\rho_0^2}c_s^4}\right).$$

3. SOLUTION ALGORITHM

The forward Laplace-Fourier transformations with respect to a time variable t and a spacious variable x_1 are used as a tool to convert the initial difficult hyperbolic two-dimensional boundary PDE problem (1)-(6) into a simpler boundary ODE problem with respect to one spacious variable x_2. The ODE problemconsists of six differential equations with respect to x_2, which can be presented in the normal form as follows:

$$\hat{w}' = B_1\hat{w} + B_2, \quad (7)$$

where $\hat{w} = \hat{w}(\hat{u}_1, \hat{u}_2, \hat{v}_2, \hat{\sigma}_{12}, \hat{\sigma}_{22}, \hat{p})^T$. The system (7) is solved using a spectral method.The two remaining solution functions are expressed in terms of the other six solutions as follows:

$$\hat{v}_1 = -\frac{1}{s\rho_0} ik_1\hat{p} + \frac{\hat{F}_1}{s} \quad (8)$$

$$\hat{\sigma}_{11} = -\frac{1}{ik_1}\frac{\partial\hat{\sigma}_{12}}{\partial x_2} - \frac{\rho_{0,s}}{\rho_0}\hat{p} - \frac{s\rho_{0,s}}{ik_1}\hat{u}_1 + \frac{\rho_{0,s}}{ik_1}\hat{F}_1 \quad (9)$$

The spectrum of the matrix B_1 in (7) is computed using MATLAB software and consists of three pairs of mutually opposite eigenvalues as follows:

$$Sp(B_1) = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ -\tau_1 \\ -\tau_2 \\ -\tau_3 \end{pmatrix}$$

The components of the spectrum $Sp(B_1)$ is computed in a symbolic form and depend on physical values of elastic modules K, α, μ and parameters, ρ_0, $\rho_{0,l}$, $and \rho_{0,s}$, characterizing the porous medium, which are calculated using the physical input parameters $\rho_{0,s}^f$, $\rho_{0,l}^f$, d_0 of the porous matrix and a saturating liquid and c_{p1}, c_{p2}, c_s the velocities of seismic waves propagating the porous medium. It was proved by the numeric experiments, that there are such values of parameters for which all six eigenvalues τ_k, $k=1,\ldots,6$ of the spectrum $Sp(B_1)$ are the distinct values, which is the important fact for the possibility to express the solution functions in the explicit form. And the matrix of eigenvectors denoted as $P=(z_1\,|\,z_2\,|\,z_3\,|\,z_4\,|\,z_5\,|\,z_6)\in M_{6\times 6}(\square)$ is such that $B_1 z_k=\tau_k z_k$ $k=1,\ldots,6$. The expressions of the elements of the eigenvectors matrix are too long algebraic expressions, and the relevant MATLAB code of computing it is included in the software complex for solving this dynamic problem.

The whole algorithm used to solve this difficult hyperbolic two-dimensional time-dependent problem (1)-(6)is presented in Figure 1:

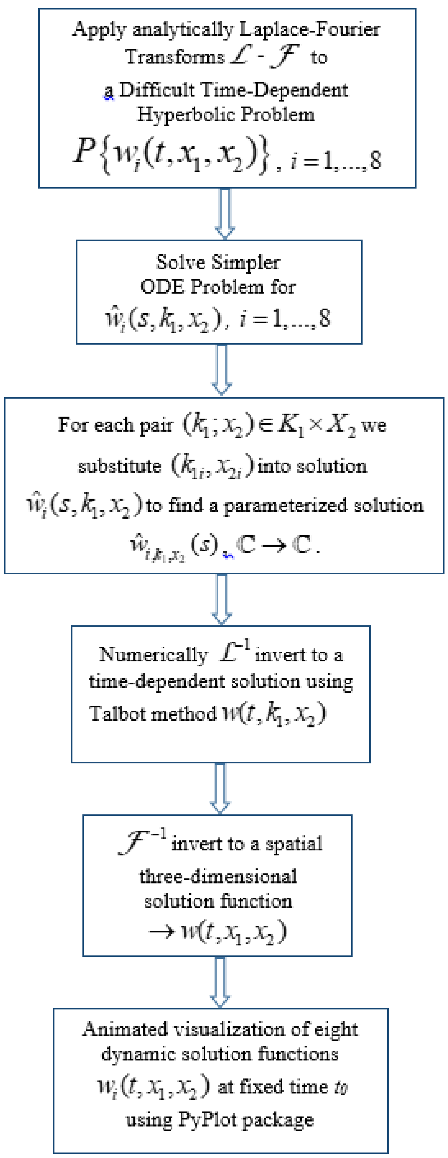

Figure 1: Solution algorithm.

It is shown in (17) that the analytical solution can be expressed in the following form:

$$\begin{aligned} w_m(t,x_1,x_2) = \int_0^{\infty}\int_{-\infty}^{\infty} \Big\{ -\sum_{k=1}^{3} z_{mk} e^{\tau_k x_2}\Big[\big(\mathrm{i}\,p_{k3}d_3+p_{k4}d_4\big)\times \\ \times\int_{x_2}^{\infty} e^{-\tau_k l}\hat{F}_1(l)dl + p_{k6}\rho_0\int_{x_2}^{\infty} e^{-\tau_k l}\hat{F}_2(l)dl\Big]+ \\ +\sum_{k=4}^{6} z_{mk} e^{-\tau_{k-3}x_2}\Big[-\sum_{j,q=1}^{3} p_{kj}r_{jq}\Big(\big(\mathrm{i}\,p_{q3}d_3+p_{q4}d_4\big)\times \\ \times\int_0^{\infty} e^{-\tau_q l}\hat{F}_1(l)dl + p_{q6}\rho_0\int_0^{\infty} e^{-\tau_q l}\hat{F}_2(l)dl\Big)+ \\ +\big(\mathrm{i}\,p_{k3}d_3+p_{k4}d_4\big)\int_0^{x_2} e^{\tau_{k-3}l}\hat{F}_1(l)dl+ \\ +p_{k6}\rho_0\int_0^{x_2} e^{\tau_{k-3}l}\hat{F}_2(l)dl\Big]\Big\}dk_1\,ds, \end{aligned} \tag{10}$$

where $m = 1,\ldots,6$

Let's denote I_k the following integral:

$$I_k=\int_0^{\infty} e^{-\tau_k l}\hat{F}_1(l)dl \tag{11}$$

which we can first solve analytically for a particular kind of function to save a lot of computational resources for computing (10).

Table 1 Physical parameters

$\rho_{0,s}^{f}$	1400.0	Kg/m^3
$\rho_{0,l}^{f}$	997.0	Kg/m^3
c_{p_1}	2000.0	m/s
c_{p_2}	1300.0	m/s
c_s	1300.0	m/s
d_0	0.3	

$$F(t,x_1,x_2)=e^{-t^2-x_1^2-x_2^2},\ (t,x_1,x_2)\in[0,\infty)\times\square\times[0,\infty). \tag{12}$$

To compute numerically the solutions(10) we take the valuesof the input parameters asstated in Table 1 and a form of a sample external force function (12) satisfying the resolvability conditions of the problem:

We first apply Laplace-Fourier transform to function $F(t,x_1,x_2)$ in (12).Then we substitute the resulted function $\hat{F}(s,k_1,x_2)$ into integral I_k (11)for further approximationof the double integrals in solution functions (10).We can first solve I_k analytically to save computational resources as follows:

$$\int_0^\infty e^{-\tau_q l}\hat{F}_1(l)dl = \frac{\pi}{2}erfc(\frac{s}{2})e^{\frac{s^2}{4}-\frac{k_1^2}{4}}\int_0^\infty e^{-\tau_k l}e^{-l^2}dl = $$
$$= \frac{\pi^{3/2}}{4}erfc(\frac{s}{2})erfc(\frac{\tau_k}{2}),$$

where $erfc$ is complimentary error function.

4. SIMULATION RESULTS

Visualizationof the solution functions is done in animated 3-dimensional format using PyPlot packages. While for the purposes of the paper the sample graphs of the solution functions are presented for two differentfixed time values $t=1$(when n=32,where n is a number of nods), and for $t=53$(when n=64, where n is a number of nods) as follows:

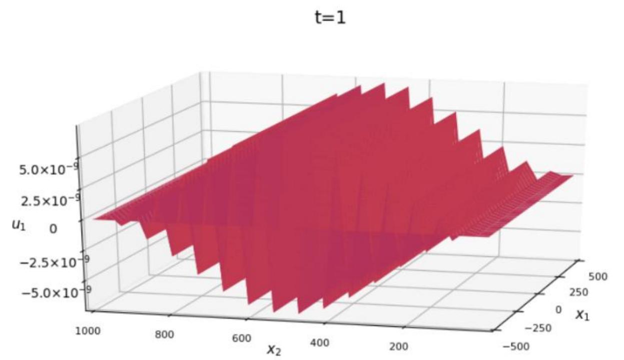

Figure 2: Solution function $u_1(t,x_1,x_2)$ at $t=1s$

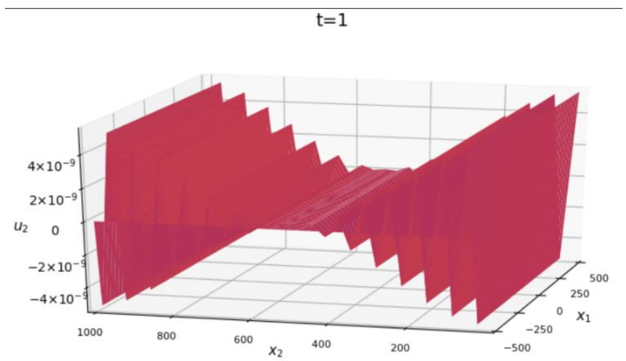

Figure 3: Solution function $u_2(t,x_1,x_2)$ at $t=1s$

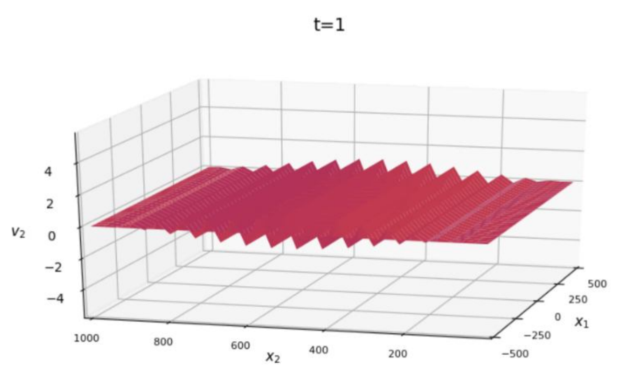

Figure 4: Solution function $v_2(t,x_1,x_2)$ at $t=1$s

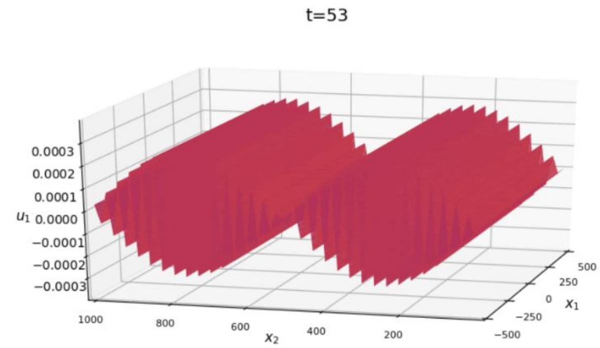

Figure 5: Solution function $u_1(t,x_1,x_2)$ at $t=53s$

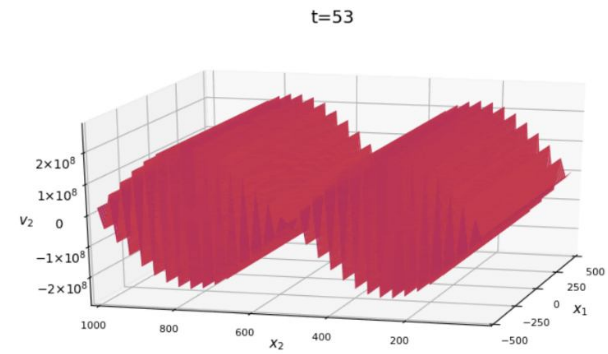

Figure 6: Solution function $v_2(t,x_1,x_2)$ at $t=53s$

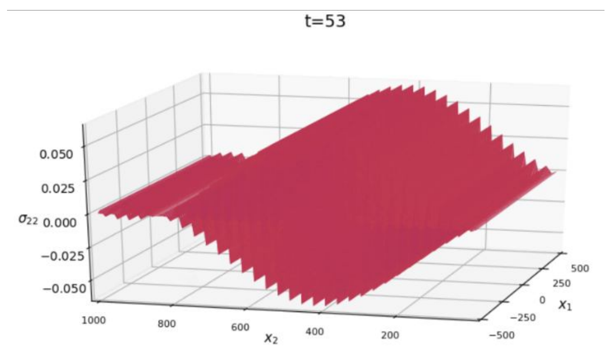

Figure 7: Solution function $\sigma_{22}(t,x_1,x_2)$ at $t=53s$

It can be seen on Figures 1-6 that the solution functions satisfy the boundary conditions, i.e. threeof them are fading to zero at $x_2=0$ and all in infinity, when $x_2\to\infty$.

5. CONCLUSION

Computer simulation and visualization presented in this paper have demonstrated that the solution functions expressed in (10) have been efficiently computed using Matlab, Julia, and PyPlot packages. Computed solution functions have been obtained following the steps of the algorithm shown in Figure 1 and describewave propagation processes in the complex porous media saturated with liquid and satisfy the necessary boundary conditions. Computations are performed using testing input data and have shown the stability of the developed algorithm for the modeled environment.

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REFERENCES

1. A.S.Berdyshev, Kh.Kh.Imomnazarov, Jian-Gang Tang,andA.Mikhailov. **The Laguerre spectral method as applied to the numerical solution of a two-dimensional linear dynamic seismic problem for porous media,** *Open Comput.Sci.,* 6, pp.208-212, 2016.
2. K.E. Sorokin, Kh.Kh. Imomnazarov. **Numerical Solving of the Liner Two-dimensional Dynamic Problem in Liquid-filled Porous Media,** *Journal of Siberian Federal University, Mathematics&Physics*, 3(2), pp.256-261, 2010 (in Russian).
   К.Э. Сорокин, Х.Х. Имомназаров. **Численное решение линейной двумерной динамической задачи для пористых сред,** *Journal of Siberian Federal University. Mathematics & Physics*, 3(2), сс. 256–261, 2010.
3. V.N. Dorovsky, Kh.Kh. Imomnazarov. **Mathematical model for the movement of a conducting liquid through a conducting porous medium,** *Mathl. Comput. Modelling*, Printed in Great Britain, Vol. 20, No.7, pp.91-97, 1994.
4. Kh.Kh. Imomnazarov, Sh.Kh. Imomnazarov, M.M. Mamatqulov, E.G. Chernykh. **The fundamental solution of the stationary two-velocity hydrodynamics equation with one pressure,** *Bull. Of the Novosibirsk Computing Center, series: Mathematical Modeling in Geophysics*, Novosibirsk, No.17, pp.5-12, 2014.
5. A.Meirmanov. ***Mathematical Models for Poroelastic Flows***, Berlin: Springer, Vol. 1 of Atlantis Studies in Differential Equations, 2014.
6. A.S.Berdyshev, Kh.Kh.Imomnazarov. ***Direct and inverse problems for the system of equations of the theory of the two-velocity continuum,*** MES RK, AbaiKazNPU, IICT. -Almaty, pp. 153, 2017 (in Russian).
   А.С. Бердышев, Х.Х. Имомназаров. ***Прямые и обратные задачи для системы уравнений теории двухскоростного континуума,*** МОН РК, КазНПУ им Абая, ИИВТ. -Алматы, -153с. 2017.
7. A.S. Berdyshev, Kh.Kh. Imomnazarov, A.E. Kholmuradov, **Optimization metod of solving a one-dimensional inverse poroelasticity problem,** *Siberian electronic mathematical proceedings*. Vol 12, pp. 284-293, 2015, http:semr.math.nsc.ru/v12/html
   А.С. Бердышев, Х.Х. Имомназаров, А.Е. Холмурадов. **Оптимизационный метод решения одной одномерной обратной задачи пороупругости** *Сибирские электронные математические известия*, Т 12, сс.284-293,http:semr.math.nsc.ru/v12/html, 2015.
8. A.S.Berdyshev, Kh.Kh.Imomnazarov, Jian-Gang Tang,S.Tuchieva. **The symmetric form of poroelasticity dynamic equations in terms of velocities, stresses, and pressure,** *Open Engineering formerly Central European Journal of Engineering.* Vol. 6, Issue 1. pp. 322-325, (2016).
9. A.S.Berdyshev, D.N.Bliyeva **Application of Laplace transformation to reduce differential system of poroelasticity equations to an elliptic system of equations,** *Proceeds of the Scientific IICT Conference Modern problems of Computer Science and Computational Technologies*, Almaty, pp. 80-81, June 2-5, 2018.
10. A.S.Berdyshev, D.N.Bliyeva. **On one study method of a mathematical model based on poroelasticity equations dynamic system,** *VestnikKazNITU*, - №3, pp. 527-533, 2019 (In Russian)
    Бердышев А.С., Блиева Д.Н**. Об одном методе исследования математической модели динамической системы уравнений пороупругости,** *Вестник КазНИТУ им.Сатпаева*, № 3, сс. 527-533, 2019.
11. A.S.Berdyshev, D.N.Bliyeva. **Construction of fundamental solution system for dynamic poroelasticity problem using Laplace-Fourier transforms**,*in Proceeds of IV International Conf. Computer Science and Applied Mathematics*, Almaty, pp. 23-31, Sep 25-29, 2019.
12. I.I.Frenkel. **On the theory of seismic and seismoelectrical phenomena in wet soil**, *Izvestia AN SSSR. Ser.Geogr. and Geophys.*,Vol.8, No.4, pp.133-150,1944(In Russian).
    Я.И. Френкель. **К теории сейсмических и сейсмоэлектрических явлений во влажной почве,** *Изв. АНСССР, Сер.геогр.и геофиз.*, Т.8, №4, сс. 133-150, 1944.
13. M.A.Biot. **Theory of propagation of elastic waves in fluid-saturated porous solids,** *J. Acoustic. Soc. Amer.*, V. 28. pp. 168 - 186, 1956.
14. V.N. Dorovskiy. **Continuous filtration theory** *Geology and Geophysics*, 1989 (7), p 39-45 (in Russian).
    ДоровскийВ.Н.**Континуальная теория фильтрации,** *Геологияи геофизика*, №7, сс. 39-45, 1989.
15. V.N.Dorovskiy, Y.V.Perepechko, E.I.Romenskiy, **Wave processes in saturated elastically deformable medium,***Physics of combustion and explosion,* No 1, pp.100-111,1993 (In Russian).
    ДоровскийВ.Н., ПерепечкоЮ.В., РоменскийЕ.И., **Волновые процессы в насыщенных пористых упругодеформируемых средах**, *Физика горения и взрыва,* №1, сс. 100-111, 1993.

16. A.M.Blokhin, V.N.Dorovsky***Mathematical Modelling in the Theory of Multivelocity Continuum***, Nova Sci., New York, MathSciNet, 1995.
17. A.S.Berdyshev, D.N.Bliyeva A.Cabada, F.Fernandez. **Sufficient conditions to obtain the exact expression of the solution of the dynamic poroelasticity problem. A numerical approach.**unpublished.
18. **Raad N.Butris**, **Hojeen M.Haji**. **Periodic solutions for nonlinear systems of integro-differential equations of Volterra- Friedholm type** *International Journal of Advanced Trends in Raad N.Butris et al., International Journal of Advanced Trends in Computer Science and Engineering,* Vol.8, No.5, pp. 2076 - 2088 September - October 2019.
19. R.N. Butris and R.F. Taher. **Periodic Solution of Integro-Differential Equations Depended on Special Functions with Singular Kernels and Boundary Integral Conditions**, *International Journal of Advanced Trends in Computer Science and Engineering,* Volume 8, No.4, pp. 1509-1520, September - October 2019,