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Stability in Delay Functional Differential Equation established using the BANACH Fixed Point Theorem

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ABSTRACT

In this paper, we using the Banach's fixed point theorem to prove the asymptotic stability of delay functional differential equation and how to obtain generalizations forms of some studied results in [11,12]. Specifically relative stability was studied; whereas the asymptotic stability in our study investigated with generalization of delay results of the functional differential equation.

Key words : Nonlinear Differential Equation, asymptotical stability, Delayed Functional Differential Equation, Banach theorem, fixed point..

1. INTRODUCTION

A. In our work, we consider a nonlinear scalar delay differential equation with variable delays and give some new conditions for the boundedness and stability by fixed point theory. A stability theorem with necessary and sufficient conditions is proved. The fixed point theory can be applied immediately to study an equation when solutions are behold at a particular interval. There are many doings achieved on the specific conduct of mentioned differential equations, According to fixed point theories, readers can judge Becker & Burton's books or papers [1], Burton [2-7], Burton & Furumochi [8, 9], Eberhard [10], Zhang [11, 12], Tunç and Mohammed [13-15].

B. Achieving the results of stability by using the fixed point theory can sometimes provide better conditions for zero solution of Lyapunov methods.

C. The advantages of this specific method have been achieved by contraction mapping methods that require intermediate conditions of vector syndrome using the chosen variation of the parameters of the type parameters to reflect the basic equation to the integrated form.

As is known in the theory of differential equations, the common way to prove solutions is through fixed point methods. However, recently, the principle of deflation mapping has been used to obtain additional properties of the solution, attractive solutions to balance, and not only the presence of curves of this solution, as usual in the theory of classical differential equation. We will explain how to use Banach's theory of fixed point in the asymptotic stability of nonlinear differential equations; also obtain appropriate generalizations and robust forms for some results in [11, 12]. Specifically, non-asymptotic stability is achieved, while we will discuss how to achieve asymptotic stability as well as stability by making a simple observation, as well as generalizing the results of previous non-asymptotic stability to systems of functional differential equations, and not only to functional differential equations Numerical as in the paper mentioned. This raises the question of how much this particular method can afford us, and what are the limitations of this technique. We will refer to the important limitation of the fixed point theory on the uniqueness of solutions only within the complete metric space area where they are not specified. If the metric space onto which the contraction mapping principle is applied is very small, we do not get a satisfactory result. We will discuss this in detail below.

2. SOME BASIC CONCEPT

2.1: Banach Contraction Mapping Principle [11, 12].

Let (X,d) be a complete metric space. A map $P: X \to X$ is a contraction if there exists a non negative number $\rho \leq 1$ such that

$$d(P(x),P(y)) \leq \rho d(x,y).$$

P is a strict contraction if $\rho < 1$.

A point $x \in X$ is called a fixed point of P if $P(x) = x$.

Theorem 2.2: (Contraction Mapping Principle). If $\phi:(S,d) \to (S,d)$ is a contraction mapping and if (S,d) is a complete metric space, then ϕ has a unique fixed point; that is, there is a unique $s^* \in S$ such that $\phi(s^*) = s^*$ **[**11, 12**]**.

Generally, the basic idea is to give a differential equation delay model

$$\left.\begin{array}{r} x' = f(t,x) \\ x(t_0) = \emptyset \end{array}\right\} \qquad (1)$$

We are trying to build a reflective schema (1). In other words, we are doing some integration operations like this

$$x(t) = a(t) + \int_{t_0}^{t} f(t,\emptyset,s,x(.))ds. \qquad (2)$$

The solution will be given to the functional differential equation through the fixed point theory mapping.

$$(Py)(t) = a(t) + \int_{t_0}^{t} f(t,\emptyset,s,y(.))ds.$$

In this way we decide that P-mapping is actually a viable representation by proving a solution of (1). Finally, the integrated solution format will provide us with stability assistance of (1). In addition, the complete metric space M will provide us with the benefits of asymptotic stability. And the M plays an important role in this, as we will see in the future.

Definition 2.3: The solution $x_\phi(t)$ of equation (1) is called stable, if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$, such that from the inequality $|\phi(t) - \psi(t)| < \delta(\varepsilon)$ on the initial set, there follows $|x_\phi(t) - x_\psi(t)| < \varepsilon$ for all $t \geq t_0$, where $\psi(t)$ is any continuous initial function [13- 15] .

Definition 2.4: The solution of equation (1) $x_\phi(t)$ is called asymptotically stable, if it is stable and $\lim_{t\to\infty} |x_\phi(t) - x_\psi(t)| = 0$, for all continuous initial functions $\psi(t)$ [14- 15] .

3. GENERAL RESULT FOR A NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION

The scalar delayed differential equation is study

$$x'(t) = -A(t)x(t) + f(t,x(t)) \qquad (3)$$

(1)Where $A: \mathbb{R}_+ \to \mathbb{R}$ and $f: \mathbb{R}_+ \times BC \to \mathbb{R}$ are continuous where $BC = \{\emptyset \in C[\mathbb{R}_-,\mathbb{R}]: \emptyset \text{ bounded}\}$. We mean by $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty,0]$ respectively, the continuous function $x: \mathbb{R} \to \mathbb{R}$ denoted by $x(t)$, $x(t(\theta)) \coloneqq x(t+\theta)$, for $\theta \in \mathbb{R}_-$.

We note that the theorem has a necessary and adequate condition for asymptotic stability in [11, 12]. We will focus only on the prevailing conditions to stabilize this work. In the above result, it indicates that zero is asymptotically stable if and only if $\int_0^t A(s)ds \to \infty$ as $t \to \infty$.

Through analysis equation (3), realize that we can provide an integrated formula equivalent to this problem by doing something similar to what we do when we solve ODE linear order of first order, if solution of (3) exists, after we multiply it by the integrating factor:

$$\mu(t) \coloneqq e^{\int_{t_0}^{t} A(s)ds}$$

To find

$$\frac{d}{dt}\big(x(t)\mu(t)\big) = \mu(t)f(t,x(t))$$

$$x(t)\mu(t) - x(t_0) = \int_{t_0}^{t} e^{\int_s^t A(u)du} f(s,x(s))ds,$$

Within the initial condition of the differential differential equation

$$x(t) = \emptyset(0)e^{-\int_{t_0}^{t} A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s,x(s))ds,$$

Thus, we have an integral equivalent expression of the solution $x(t)$. This indicates that we are define the following mapping P defined on S:

$$(Px)(t) \coloneqq \begin{cases} \emptyset(t-t_0) \quad if \quad t \le t_0 \\ \emptyset(0)e^{-\int_{t_0}^{t} A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s,x(s))ds, if \quad t \ge t_0 \end{cases}$$

In order to apply fixed point theory, we must prove that P maps S to itself, $Px: \mathbb{R} \to \mathbb{R}$ is continuous, $Px(t_0) = \emptyset$, $\emptyset \in C(\delta_0)$, and $\|x_s\| \le L$.

In [11, 12] we have the following conditions:

(1) $\lim f_{t\to\infty} \int_0^t A(s)ds > -\infty$.

(2) $\int_0^t e^{-\int_{t_0}^t A(u)du} b(s)ds \le \alpha < 1$ for all $t \ge 0$.

(3) $|f(t,\emptyset) - f(t,\psi| \le b(t)\|\emptyset - \psi\|$, $for\ all\ \emptyset, \psi \in C(L), and\ f(t,0) = 0$.

(4) $\int_0^t A(s)ds \to \infty \quad as \quad t \to \infty$,

Then the zero solution of (3) is asymptotic stabile.

(5) $\forall \varepsilon > 0$ and $t_1 \ge 0$, then $\exists\, t_2 > t_1$ s.t. $t \ge t_2$, and $x(t) \in C(L)$, this implise that $|f(t,x(t))| \le b(t)(\varepsilon + \|x\|^{[t_1,t]})$.

Then by using the fact $\emptyset \in C(\delta_0)$, and $\|x(s)\| \le L$ and condition (2, 3) from [11, 12], we have that

$$|(Px)(t)| \le \left|\emptyset(0)e^{-\int_{t_0}^t A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s,x(s))ds\right|$$

$$\le \delta_0 e^{-\int_{t_0}^t A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} b(s)\|x(s)\|ds$$

$$\le \delta_0 K + L\int_{t_0}^{t} e^{-\int_s^t A(u)du} b(s)ds$$

$$\le \delta_0 K + L\alpha \le L$$

Now we will show that $(Px)(t) \to 0\ as\ t \to \infty$, since $x(t) \to 0\ as\ t \to \infty\ and\ \exists\, t_1 > t_0$, s.t. $|x(t)| < \varepsilon \quad \forall\ t \ge t_1$, then by condition (5) in [11, 12], $\exists\ t_2 \ge t_1$ s.t. $t \ge t_2$ this implies that

$$|f(t,x(t))| \le b(t)(\varepsilon + \|x\|^{[t_1,t]}) \text{ for } t \ge t_2$$

$$\left|\int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s,x(s))ds\right|$$

$$\le \left|\int_{t_0}^{t_2} e^{-\int_s^t A(u)du} f(s,x(s))ds\right| + \left|\int_{t_2}^{t} e^{-\int_s^t A(u)du} f(s,x(s))ds\right|$$

$$\le \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} |f(s,x(s))|ds + \int_{t_2}^{t} e^{-\int_s^t A(u)du} |f(s,x(s))|ds$$

$$\le \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} b(s)\|x(s)\|ds + \int_{t_2}^{t} e^{-\int_s^t A(u)du} b(s)(\varepsilon + \|x\|^{[t_1,t]})ds$$

Since $\|x\|^{[t_1,t]} \le \varepsilon$ for $\ge t_2$, then we get

$$\le \|x(t)\| \int_{t_0}^{t_2} e^{-\int_s^t A(u)du} b(s)ds + \int_{t_2}^{t} e^{-\int_s^t A(u)du} b(s)(2\varepsilon)ds$$

$$\le L\int_{t_0}^{t_2} e^{-\int_s^{t_2} A(u)du - \int_{t_2}^t A(u)du} b(s)ds + 2\varepsilon\alpha$$

$$\le \alpha L e^{-\int_{t_2}^t A(u)du} + 2\varepsilon\alpha$$

By condition (4) in [11, 12] $\exists\, t_3 \ge t_2$,such that

$$\delta_0 e^{-\int_{t_0}^{t} A(u)du} + L\, e^{-\int_{t_2}^{t} A(u)du} < \varepsilon$$

This two estimate implies that $t \geq t_3$, and

$$|(Px)(t)| \leq \left| \emptyset(0)e^{-\int_{t_0}^{t} A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s, x(s))ds \right|$$

$$\leq \delta_0 e^{-\int_{t_0}^{t} A(s)ds} + \alpha L e^{-\int_{t_2}^{t} A(u)du} + 2\alpha\varepsilon < 3\varepsilon$$

This prove that $(Px)(t) \to 0 \ as \ t \to \infty$, and $Px \in S$ for $\forall \ x \in S$, and $P: S \to S$ is well define, now to proving P is a contraction mapping on S, let $x, y \in S$:

$$|(Px)(t) - (Py)(t)| \leq$$
$$\int_{t_0}^{t} e^{-\int_s^t A(u)du} |f(s, x(s)) - f(s, y(s))| ds$$

$$\leq \int_{t_0}^{t} e^{-\int_s^t A(u)du} b(s) \|x(s) - y(s)\| ds$$

$$\leq suprum\, |x(s) - y(s)| \int_{t_0}^{t} e^{-\int_s^t A(u)du} b(s)ds$$

$$\leq \alpha\rho(x, y)$$

By Contraction Mapping Principle theorem $\exists$ unique fixed point $x \in S$ that solve equation (3) for each $\emptyset \in C(\delta_0)$, and we have $x(t) = x(t, t_0, \emptyset)$ converge to zero as $t \to \infty$.

Let $\varepsilon > 0, \ \varepsilon < L$ are given we will find $\delta < \varepsilon$ s.t. $\delta K + \alpha\varepsilon < \varepsilon$, then

$$x(t) = x(t, t_0, \emptyset) = \emptyset(0)e^{-\int_{t_0}^{t} A(s)ds} + \int_{t_0}^{t} e^{-\int_s^t A(u)du} f(s, x(s))ds,$$

We prove that $|x(t)| < \varepsilon$,$\forall \ t \geq t_0$, and $\exists \ T > t_0$ notice that $x(t_0) < \delta < \varepsilon$ fot $t_0 \leq s < T$. But $|x(T)| = \varepsilon$, then

$$|x(T)| \leq$$
$$\delta e^{-\int_{t_0}^{T} A(s)ds} + \int_{t_0}^{T} e^{-\int_s^T A(u)du} b(s)\|x(s)\|ds,$$
$$\leq \delta K + \alpha\varepsilon < \varepsilon \,,$$

Which is contradiction to the definition of T. This means that there T is not exist and $x(t) < \varepsilon$ for $\forall \ t \geq t_0$, then the zero solution of equation (3) is asymptotic stable.

CONCLUSION

In this paper, we consider a nonlinear scalar delay differential equation with variable delays and give some new conditions for the boundedness and stability by means of Krasnoselskii's fixed point theory. A stability theorem with a necessary and sufficient condition is proved.

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