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The Reliability of the Recursive Corona Product Network

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ABSTRACT

The number of spanning trees is a determinant factor of dynamic properties of networks, such as their reliability. The well-known algebraic method computes this number, is the Matrix-Tree Theorem. However, the calculation using this method is very tedious and time consuming, in particular for large networks. For this reason there is so much interest to investigate explicit formula computing this number for relevant infinite graph families. In this paper, we aim to give an exact analytic expressions for the number of spanning trees in a small world network such as the recursive corona product G_n^k, which is similar to those existed in real life networks. In addition, we aim at calculating the asymptotic spanning tree entropy of this product graph.

Key words: Asymptotic Entropy, Corona product, Graph complexity, Small world network, spanning tree.

1. INTRODUCTION

Enumerating spanning trees of graph is one of the most studied problem in several areas such as mathematics [1], [7] physics [8], and computer science [10]. The number of spanning trees is an important network parameter, it is very useful in many practical areas such as artificial neural networks. To estimate the reliability level of this complex system, the number of spanning trees is good factor of Systems Performance [11].

The number of spanning trees is often called the complexity of the graph G, denoted by (G). Methods computing the number of spanning trees in a graph have been investigated for more than 200 years. The best known algebraic method that calculates this number is the Matrix-Tree Theorem which expresses the number of spanning trees as a co-factor matrix determinant of the Laplacian matrix of the graph G [1]. However the stated algorithm is significantly complex and poses serious issues for large graphs. For this reason, there has been much interest in finding efficient alternate methods in order to give explicit expressions of this number for some graph families such as Sierpinski gaskets [14] grids[12] and lattices [13], [15]. In this frame of reference, several methods were developed based on different principles. But these methods require a lot of calculus and are too intricate. The authors developed methods based on Chebyshev Polynomials in [6] to count the complexity of corona product of some special graphs that need a lot of algebraic calculation. In our case, we propose an efficient combinatorial method, to investigate an exact formula for the number of spanning trees of an infinite family of outer planar graphs, such as small-world network that has several interesting properties. These properties of small-world can be found in networks associated namely to social network [16, 17]. In this work we give an explicit analytic expression computing the number of spanning trees in the recursive corona product.

1.1 BASIC NOTIONS AND DEFINITIONS.

Now we introduce a recursive way based on corona product of graph G_n^k (k denotes the k^{th} iteration), to construct small world networks with an exponential degree distribution. Let G_{n1} *and* G_{n2} be two graphs, n_1 and n_2 denote the number of vertices in G_{n1} *and* G_{n2} respectively. The corona product $G_{n1} \lozenge G_{n2}$ is the graph which consists of one copy of G_{n1} and n_1 copies of G_{n2}, by joining the i^{th} vertex of G_{n1} by a new edge with every vertex in the i^{th} copy of G_{n2}[1], see figure 1. The order (number of vertices) of $G_{n1} \lozenge G_{n2}$ is

$|G_{n1} \lozenge G_{n2}| = |V(Gn1)| + |V(Gn1)| * |V(Gn2)|$
$= n_1 + n_1 * n_2.$

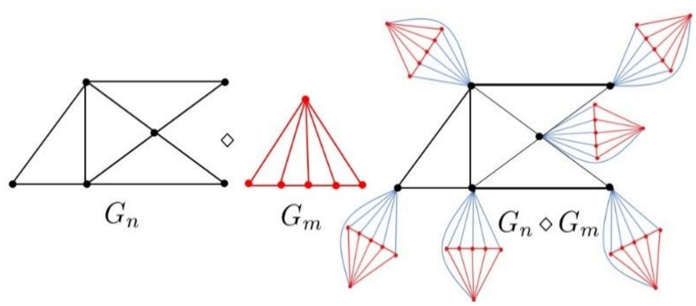

Figure 1: Planar graphs G_n , G_m and the corona product $G_n \lozenge G_m$

Thus, the recursive corona graph Gkn is defined as follow, the k^{th} generation of recursive corona graph is the graph obtained from the corona product of the previous generation $(k-1)^{th}$ of $G_n^{k-1}$and G_n(see figure 2) $G_n^k = G_n^{k-1} \lozenge G_n$ with $G_n^0 = G_n$. The number of spanning trees of an infinite graph like small-world network G_n^k has asymptotic exponential growth. That is why the entropy of spanning trees $\rho(G_n^k)$ is used to determine level of reliability of networks rather than the number of spanning trees [19]. The entropy can be used to provide a natural measure of the rate of growth. $\rho(G_n^k)$ is defined [18] as the limiting value:

$$\rho(G_n^k) = \lim_{|V_{(G_k)}|\to+\infty} \frac{\log\tau(G_n^k))}{|V_{(G_n^k))}|}$$

The finite value of $\rho(G_n^k)$ characterizing the network structure, is a quantity of physical interest.

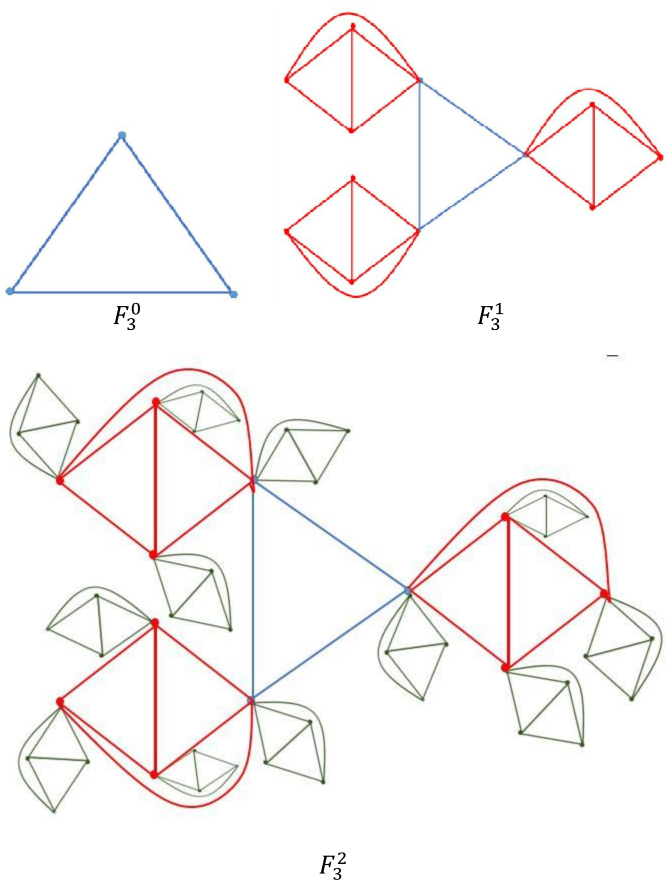

Figure 2: Generation Process of the recursive corona product, showing the first three iteration of F_3^k

1.2 PRELIMINARY

In this section we introduce some theorems which will provide basis to our work.

Theorem 1: Let G_{n1} be a planar graph and G_{n2} an outer planar graph. So, the number of spanning trees in the corona product graph of G_{n1} and G_{n2} is given by the following formula [2]:

$$\tau(Gn1 \lozenge Gn2) = \tau(G_{n1}) * \tau(P_1 \lozenge G_{n2})^{n1}$$

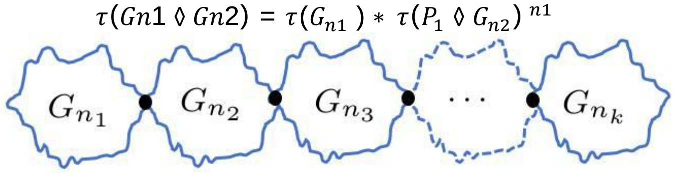

Figure 3. The map $G_{n,k}$

Theorem 2 : Let $G_{n,k}$ be a map formed of k sub-graphs as shown in Figure 3($G_{n,k}= G_{n1} \bullet G_{n2} \bullet G_{n3} \bullet\bullet\bullet G_{nk}$), in such way that each adjacent pair sub-graph has an articulation vertex in common. So, the number of spanning trees in $G_{n,k}$ is given by [3]:

$$\tau(G_{n1} \bullet G_{n2} \bullet G_{n3} \bullet\bullet\bullet G_{nk}) = \prod_{i=1}^{k} \tau(G_{n_i})$$

Theorem 3: Let F_n be a fan graph (See Figure 4(a)). The number of spanning trees in the fan graph is given by [4]:

$$\tau(F_n) = \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right)$$

Theorem 4 (Euler's Formula) Let G be a connected planar graph, and let n, m and f denote, respectively, the numbers of vertices, edges, and faces in a planar graph of G. Then: $n - m + f = 2$ [1].

2. RESULTS

In this section we make use of our method to calculate the number of spanning trees in the small-world network F_n^k (recursive corona product of the fan graph F_n), then we give an explicit formula that calculates its asymptotic spanning tree entropy (F_n^k). Then we treat the general case investigating the complexity (G_n^k) of recursive corona product G_n^k and its asymptotic spanning tree entropy $\rho(G_n^k)$. At first, In order to compute (G_n^k) and make sure that G_n^k is a planar graph, we calculate the number of vertices, number of edges and number of faces in G_n^k. Let G_n be a planar graph, n, m, f denote the number of vertices, number of edges and number of G_n faces in G_n.

$$|(V(G_n)| = n, \quad |E(G_n)| = m, \quad |F(G_n)| = f).$$

Lemma 1: The number of vertex, number of edges and number of faces in Gkn are given by:

$$|V(G_n^k)| = n(n+1)^k$$
$$|E(G_n^k)| = (n+m)(n+1)^k - n$$
$$|F(G_n^k)| = (n+f-2)(n+1)^k - n + 2$$

Proof 1: According the definition of corona product, The number of edges in G_n^k is given by :

$$|E(G_n^k)| = |E(G_n^{k-1})| + |V(G_n^{k-1})|(n+m)$$
$$|E(G_n^{k-1})| = |E(G_n^{k-2})| + |V(G_n^{k-2})|(n+m)$$
$$\vdots$$
$$|E(G_n^1)| = |E(G_n^0)| + |V(G_n^0)|(n+m)$$

by adding these k equations we obtain,

$$|E(G_n^k)| = |E(G_n^0)| + |V(G_n^0)|(n+m) + |V(G_n^1)|(n+m) + \cdots + |V(G_n^{k-1})|(n+m)$$
$$|E(G_n^k)| = m + n(1+n)^0(n+m) + n(1+n)^1(n+m) + \cdots + n(n+1)^{k-1}(n+m)$$

Hence the result. To prove the others formulas, we use the same method. Remark: The graph Gkn is still planar graph. in order to prove that, we use Euler Theorem.

$$|F(G_n^k)| + |V(G_n^k)| - |E(G_n^k)| = (n+f-2)(n+1)^k - n + 2 + n(n+1)^k - ((n+m)(n+1)^k - n) = (n+f-2-m)((n+1)^k + 2$$

As G_n a planar graph, $(n + f - 2 - m = 0$, that means G_n^k is a planar graph. Now we give an explicit formula to calculate the number of spanning trees (G_n) and the asymptotic entropy of the recursive corona product $\rho(G_n^k)$.

Theorem 1 The complexity of the recursive corona product of a planar graph G_n is :

$$\tau(G_n^k) = \tau(G_n) \times \tau(G_n \Diamond P_1)^{(n+1)^k - 1}$$

Proof 2:

$$\tau(G_n^k) = \tau(G_n^{k-1} \Diamond G_n) = \tau(G_n^{k-1}) \times \tau(G_n \Diamond P_1)^{n(n+1)^{k-1}}$$

$$\tau(G_n^{k-1}) = \tau(G_n^{k-2} \Diamond G_n) = \tau(G_n^{k-2}) \times \tau(G_n \Diamond P_1)^{n(n+1)^{k-2}}$$

$$\vdots$$

$$\tau(G_n^1) = \tau(G_n^0 \Diamond G_n) = \tau(G_n^0) \times \tau(G_n \Diamond P_1)^{n}$$

We multiply these equations we get:

$$\tau(G_n^k) = \tau(G_n) \times \tau(G_n \Diamond P_1)^{n} \times \tau(G_n \Diamond P_1)^{n(n+1)} \times \tau(G_n \Diamond P_1)^{n(n+1)^2} \times \cdots \times \tau(G_n \Diamond P_1)^{n(n+1)^{k-1}}$$

$$\tau(G_n^k) = \tau(G_n) \times \tau(G_n \Diamond P_1)^{\sum_{i=0}^{k-1} n(n+1)^i}$$

Hence the result.

After having an exact expression for the number of spanning trees of the graph Gkn, now we can calculate its spanning tree entropy, as defined in [18].

Theorem 2: Let G_n^k be the recursive corona product of a planar graph G_n. The asymptotic spanning tree entropy of the graph G_n^k is given by: $\rho(G_n^k) = \frac{\log \tau(G_n \Diamond P_1)}{n}$

Proof 3: The spanning tree entropy is defined as the limiting value: $\rho(G_n^k) = \lim_{|V_{(G_k)}| \to +\infty} \frac{\log \tau(G_n^k)}{|V(G_n^k)|}$

Using Theorem 1 and substituting by its expression, where $|V(G_n^k)| = n(n+1)^k$, we get :

$$\rho(G_n^k) = \lim_{|V_{(G_k)}| \to +\infty} \frac{\log \tau(G_n^k)}{|V(G_n^k)|}$$

$$\rho(G_n^k) = \lim_{|V_{(G_k)}| \to +\infty} \frac{\log \left(\tau(G_n) \times \tau(G_n \Diamond P_1)^{(n+1)^k - 1}\right)}{|V(G_n^k)|}$$

$$= \lim_{|V_{(G_k)}| \to +\infty} \frac{\log \tau(G_n)}{|V(G_n^k)|} + \left(\frac{1}{n} - \frac{1}{|V(G_n^k)|}\right) \frac{\log (\tau(G_n \Diamond P_1))}{|V(G_n^k)|}$$

Hence the result.

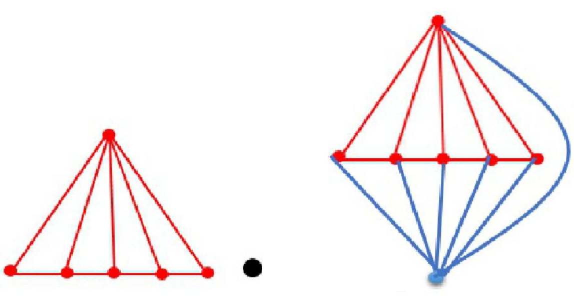

Figure 4: F_6 , P_1 *and* $F_6 \Diamond P_1$

Theorem 3 :

Let F_n be a fan (see Figure 4(b)), the number of spanning trees in the recursive corona product F_n^k of Fan graph F_n is given by:

$$\tau(F_n^k) = \frac{1}{\sqrt{5}}\left(\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right) \times \left(\frac{n+1}{2\sqrt{3}}\left((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}\right)\right)^{n(n+1)^{k-1}}$$

Corollary 1. Let F_n^k be the recursive corona product of the Fan graph Fn. The asymptotic spanning tree entropy of the graph $\rho(F_n^k)$ is given by:

$$\rho(G_n^k) = \frac{1}{n} \times \log \tau\left(\frac{n+1}{2\sqrt{3}}\left((2+\sqrt{3})^{n-1} - (2-\sqrt{3})^{n-1}\right)\right)$$

For n = 3. $\rho(F_3^k) = 0{,}92419624074659$. We can compare the obtained asymptotic spanning tree entropy value of the graph F_3^k with other graphs that have the same average degree. For example, the value of spanning tree entropy in the square lattice graph is 1,16624 [8], for the 3-dimensional Sierpinski graph is 1,5694 [14] and for the 3-dimensional hyper-cubic lattice is 1,6734 [13]. Although, the number of spanning trees in recursive corona product network grows exponentially, its asymptotic entropy value shows the rate of growth is lower than these graphs which have the same average degree. Therefore, this result reveals that the recursive corona product networks have a smaller number spanning trees; for that reason, their level of reliability is less than the networks cited above.

3. CONCLUSION

In this paper we give an explicit expression to calculate the number of spanning trees in recursive corona product networks by using a combinatorial approach, based on contraction and separation methods, which allows us to obtain the number of spanning trees for any graph order and any number of iterations. Then, knowing the number of spanning trees for recursive corona product networks we can find their asymptotic spanning tree entropy. In the near future, we aim to present new method to construct anther type of small world network by studying its structural properties.

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