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Computing the number of spanning trees in the Wheel multiple graph

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ABSTRACT

The number of spanning trees in a network relevant to several networks aspects, namely their topological and dynamic properties such as their reliability. Generally, this number can be obtained by the well-known Matrix Tree Theorem. However, computing the number of spanning trees in a network using this method is computationally demanding specially for a large network, which leads us to investigate the number of spanning trees in planar graphs. In this paper, we propose an efficient method to find exact formulas for counting the number of spanning trees in a general type of the multiple planar wheel graph.

Key words: multiple-edges, multiple-vertices, planar graph, number of spanning trees, k-multiple wheel graph.

1. INTRODUCTION

Non-oriented graphs are considered in this paper. If more than one edge is allowed to join two vertices, then the graph is called a multiple-graph; otherwise, it is called a simple graph. A connected graph without cycles is called a tree. Given a graph G, any sub graph which is also a tree with the same number of vertices as G is called a spanning tree of G. In graph theory, finding the number of spanning trees in a large graph is a hard problem. The number of spanning trees in the graph G denoted by τ (G), that is also called the complexity of the graph G [1].

The number of spanning trees of a graph is one of the most studied quantity in graph theory, this number appears in a several applications such as network reliability [8-10], cryptography [7], enumerating certain chemical isomers [11], and counting of the Eulerian circuits in a graph [12]. The study of this number was initiated by the physicist Kirchhoff who proposed a method called "Matrix Tree Theorem" expressing the number of spanning trees for any graph by computing the determinant of the Laplacian matrix defined by : L(G) = D(G)-A(G), with D(G) and A(G) are respectively the degrees matrix and the adjacency matrix [1]. However, in case of large graphs, this theorem becomes impractical due to the determinant of a large matrix. That is why many combinatorial methods have been developed for some graph families [13-15]. The remedy for this problem was to develop simple techniques based on suppression deletion and decomposition of the graph to sub-graphs and treat each sub-graph separately [1,2]. However, these methods concern only simple graphs containing a single edge, thus we proposed a generalization of Eisner's method to treat multiple-graphs with multiple k-edges [6]: $\tau(G) = \tau(G - E_k) + k\tau(G.uv)$.

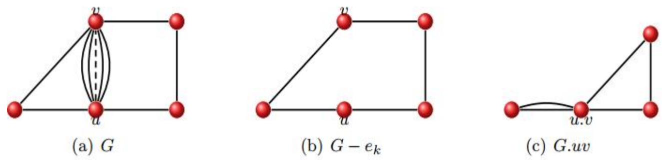


Figure. 1 : *The graphs* $G, (G - E_k), (G.uv)$

The enumeration of spanning trees for large graphs with infinite vertices is a demanding and a difficult task, thus there is much interest in obtaining closed expressions. Wherefore, many works derive formulas to calculate the complexity for some classes of simple graphs (without loops or multiple edges) Myers[4] and Haghighi[3] derives the explicit formulas $\tau(W_n)$; the number of spanning trees in wheel graphs with $n = V(W_n) - 1$:

$$\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

The authors have given an algorithm for counting the number of spanning trees in the wheel graph. In this paper, we generalize their results and provide a linear algorithm for counting the number of spanning trees of the planar graphs shown above for $n \geq 2$. In the following, we deal with multiple-edges, three type of wheels presented W_n, $W_{k,n}$ *and* $W_{k_1,k_2,n}$. We focus on the derivative of the wheel graph $W_{k_1,k_2,n}$ as is illustrated at the left of the figure 2.

The graph $W_{k_1,k_2,n}$ is obtained, by substituting each edges of the internal face in the graph W_n by a path that contains k_1 edges, and substituting each edge of the outer face in the graph W_n by ,k_2-multiple edges. So, the graph $W_{k_1,k_2,n}$ contains $n(k_1 + k_2)$ edges, $nk_1 + 1$ vertices and $n(k_2 - 1) + n$ faces. In this work we are interested to generalize the case of wheel graph to a multiple wheel graph with $(k_1 - 1)$ multiple-vertices and k_2-multiple-edges.

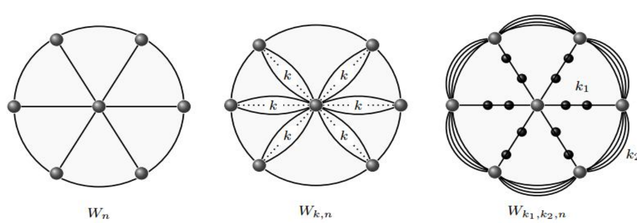


Figure 2: The wheel, k-multiple edges wheel and ($k_1$multiple vertex and k_2 multiple edges) wheel graphs.

2. PRELIMINARY NOTES

In this part we give the used theorems in our work.

Theorem 1: Let G be a graph formed by two sub graphs G_1 and G_2 that have one common vertex v ($G = G_1 \bullet G_2$). The number of spanning trees in $G = G_1 \bullet G_2$ is given by:

$$\tau(G) = \tau(G_1 \bullet G_2) = \tau(G_1) \text{x} \tau(G_2). \quad (1)$$

Theorem 2: Let G is a graph composed of two sub-graphs G_1 and G_2 ($G = G_1|G_2$) which have a common simple path P. The number of spanning trees in G is given by:

$$\tau(G_1|G_2) = \tau(G_1) \text{x} \tau(G_2) - k^2 \tau(G_1 - p) \text{x} \tau(G_2 - p) \quad (2)$$

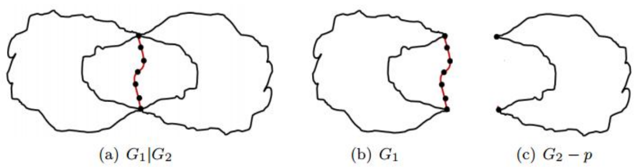


Figure 3: The graphs$G_1|G_2$, G_1 and $G_1 - p$.

Where $G_1 - p$ and $G_2 - p$ denoted the graphs obtained from G_1 and G_2 respectively after removing the simple path of length k. (i.e. P = v_1, v_2... v_k ,v_{k+1}contains k edges, as illustrated in figure 3.) [5]

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \quad (3)$$

$$\sum_{i=0}^{n} i x^{i-1} = \frac{n x^{n-1}(x-1) + 1 - x^n}{(1-x)^2} \quad (4)$$

3. COMPLEXITY OF GRAPHS WITH MULTIPLE EDGES.

3.1. THE FAN GRAPH WITH MULTIPLE EDGES.

Let $F_{k_1,k_2,i,j}$ be the fan graph with k1 multiple vertices and k_2 multiple edges, i and j denote respectively the number of paths between u and v and the sets number of multiple edges k_2 and $f_{k_1,k_2,i,j}$ denotes its complexity. In this part we enumerate spanning trees of the fan graph $F_{k_1,k_2,i,j}$with multiple vertices and edges, we consider the k_1, k_2-(multiple-edges) as is shown in Fig. 4. We found exact formulas to compute its complexity.

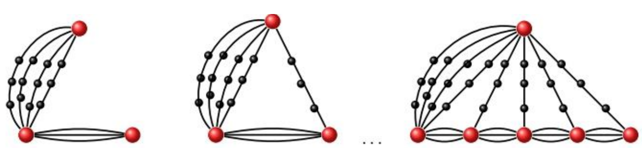


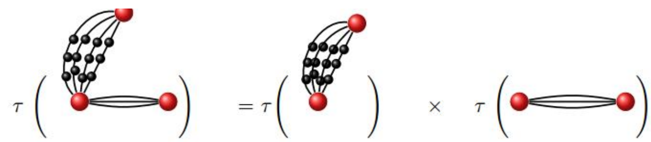
Figure 4: family of k1, k2-multiple-vertices and edges fan graphs: $F_{k_1,k_2,i,0}$; $F_{k_1,k_2,i,1}$;$F_{k_1,k_2,i,j}$.

Lemma 1:

$$\begin{cases} f_{k_1,k_2,i,0} = k_2 \text{ x } (1 + \text{i}) {k_1}^i \\ f_{k_1,k_2,i,1} = (2k_1k_2 + 1 + i(1 + k_1k_2))\, {k_1}^i \end{cases}$$

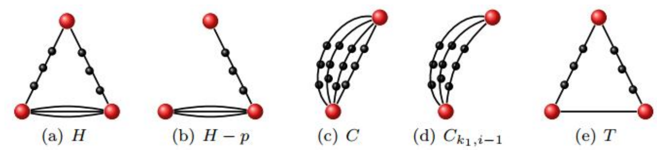
Proof:

- The graph $F_{k_1,k_2,i,0}$ has one articulation vertex and contains two sub graph connected, by using theorem 1 we get: $f_{k_1,k_2,i,0} = c_{k_1,i}$ x g_{k_2}.

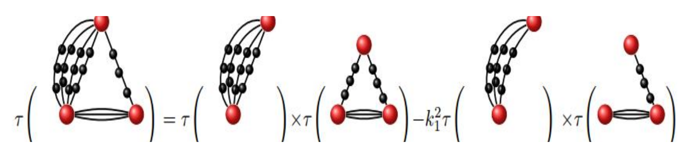


The complexity of the second one is $g_{k_2} = k_2$. The complexity of the first one is obtained by using theorem 2 we get:

$c_{k_1,i} = 2k_1 c_{k_1,i-1} - {k_1}^2 c_{k_1,i-2}$. The characteristic equation of this sequence is: $r^2 - 2k_1 r + {k_1}^2 = 0$, that has one solution $r = k_1$. Then $c_{k_1,i} = (\alpha i + \beta){k_1}^i$, by using the initial conditions we get $c_{k_1,i} = (i+1){k_1}^i$, thus the result.



- To compute the number of spanning trees in $F_{k_1,k_2,i,1}$we use theorem 2 we get:



$$f_{k_1,k_2,i,1} = \tau(C_{k_1,i}) \text{x}\, \tau(H) - {k_1}^2 \tau(C_{k_1,i-1}) \text{x}\, \tau(H - p)$$

$$\tau(H) = \tau(T) \text{x} g_{k_2} - g_{k_2 - 1} = (2k_1 + 1)k_2 - (k_2 - 1).$$

$$= 2k_2k_2 + 1.$$

$$f_{k_1,k_2,i,1} = (i+1){k_1}^i (2k_2k_2 + 1) - {k_1}^2 i {k_1}^{i-1} {k_2}^2.$$

Hence the result.

Theorem 3: The number of spanning trees in the graph $F_{k_1,k_2,i,j}$is given by:

$$f_{k_1,k_2,i,j} = \frac{{k_1}^i}{r_2 - r_1}[((r_2 - 2k_1k_2 - 1) + i(r_2 - k_1k_2 - 1)){r_1}^j +$$
$2k1k2+1-r1 +ik1k2+1-r1r2j$

Where $\begin{cases} r_1 = \frac{2k_1k_2 + 1 - \sqrt{4k_1k_2 + 1}}{2} \\ r_2 = \frac{2k_1k_2 + 1 + \sqrt{4k_1k_2 + 1}}{2} \end{cases}$

Proof:

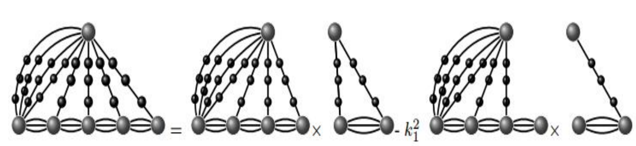


Figure 5: Complexity of $F_{k_1,k_2,i,j}$ based on Theorem 2.

To compute spanning trees of $F_{k_1,k_2,i,j}$ we use Theorem 2, wet get: $f_{k_1,k_2,i,j} = (2k_1k_2+1)f_{k_1,k_2,i,j-1} - {k_1}^2{k_2}^2 f_{k_1,k_2,i,j-2}$

The characteristic equation of this sequence is:
$r^2 - (2k_1k_2+1)r + {k_1}^2{k_2}^2 = 0.$

The solutions of this equation are:

$r_1 = \frac{2k_1k_2+1-\sqrt{4k_1k_2+1}}{2}$ and $r_2 = \frac{2k_1k_2+1+\sqrt{4k_1k_2+1}}{2}$.

Then $f_{k_1,k_2,i,j} = \alpha {r_1}^j + \beta {r_2}^j$.

Using the initial conditions of the lemma 1:

$$\begin{cases} f_{k_1,k_2,i,0} = \alpha + \beta = k_2 \times (1+i){k_1}^i \\ f_{k_1,k_2,i,1} = \alpha r_1 + \beta r_2 = (2k_1k_2 + 1 + i(1+k_1k_2))\,{k_1}^i \end{cases}$$

We solve this system we get :

$$\begin{cases} \alpha = \frac{1}{r_2 - r_1}\left((r_2 - 2k_1k_2 - 1) + i(r_2 - k_1k_2 - 1)\right){k_1}^i \\ \beta = \frac{1}{r_2 - r_1}\left((2k_1k_2 + 1 - r_1) + i(k_1k_2 + 1 - r_1)\right){k_1}^i \end{cases}$$

Hence the result.

3.2. THE WHEEL GRAPH WITH MULTIPLE EDGES.

In this part we provide an exact formula computing the number of spanning trees in the Wheel graph with multiple vertices and edges $W_{k_1,k_2,n}$, as is shown in the figure 2.

Lemma 2:

Let $W_{k_1,k_2,n}$ be the wheel graph with k_1 multiple vertices and k_2 multiple edges, and $w_{k_1,k_2,0,n}$ its complexity.

$$w_{k_1,k_2,0,n} = \sum_{i=0}^{n-1} {k_2}^i \times f_{k_1,k_2,i,n-1-i}$$

Proof: To compute $w_{k_1,k_2,0,n}$, we use deletion-contraction method of multiple edges recursively on $W_{k_1,k_2,n}$ as illustrated in Figure 6:

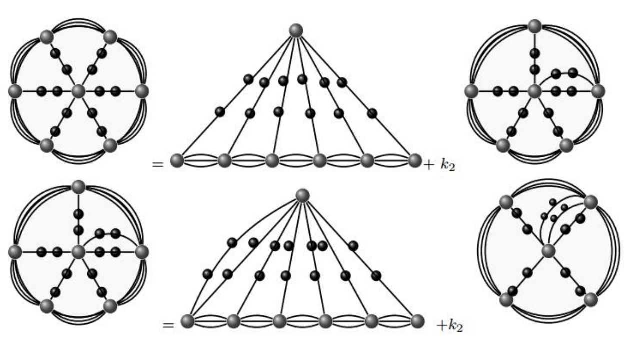


Figure 6 : Complexity of the multiple-edges wheel graph.

$$w_{k_1,k_2,0,n} = f_{k_1,k_2,0,n-1} + k_2 w_{k_1,k_2,1,n-1}$$

$$k_2 w_{k_1,k_2,1,n-1} = k_2 f_{k_1,k_2,1,n-2} + {k_2}^2 w_{k_1,k_2,2,n-2}$$

$${k_2}^2 w_{k_1,k_2,2,n-2} = {k_2}^2 f_{k_1,k_2,2,n-3} + {k_2}^3 w_{k_1,k_2,3,n-3}$$

$$\vdots$$

$${k_2}^{n-2} w_{k_1,k_2,n-2,2} = {k_2}^{n-2} f_{k_1,k_2,n-2,1} + {k_2}^{n-1} w_{k_1,k_2,n-1,1}$$

By summing previous equations we obtain the result.

Theorem 4:

The Complexity of the k_1, k_2-multiple-(vertices, edges) wheel graph is given by the following formula :

$$\begin{cases} w_{k_1,k_2,0,n} = 2(k_1k_2)^n + {r_1}^n + {r_2}^n \\ r_1 = \frac{2k_1k_2 + 1 - \sqrt{4k_1k_2+1}}{2} \\ r_2 = \frac{2k_1k_2 + 1 + \sqrt{4k_1k_2+1}}{2} \end{cases}$$

Proof:

We replace the result found in Theorem 3 in the formula of Lemma 2.

$$w_{k_1,k_2,0,n} = \sum_{i=0}^{n-1} {k_2}^i \times f_{k_1,k_2,i,n-1-i}$$

$$(r_2 - r_1)w_{k_1,k_2,0,n} = \sum_{i=0}^{n-1}(k_1k_2)^i (r_2 - 2k_1k_2 - 1){r_1}^{n-1-i}$$
$$+ \sum_{i=0}^{n-1}(k_1k_2)^i\, i(r_2 - k_1k_2 - 1){r_1}^{n-1-i}$$
$$+ \sum_{i=0}^{n-1}(k_1k_2)^i (2k_1k_2 + 1 - r_1){r_2}^{n-1-i}$$
$$+ \sum_{i=0}^{n-1}(k_1k_2)^i\, i(k_1k_2 + 1 - r_1){r_2}^{n-1-i}$$

$$(r_2 - r_1)w_{k_1,k_2,0,n} = (r_2 - 2k_1k_2 - 1){r_1}^{n-1}\sum_{i=0}^{n-1}\left(\frac{k_1k_2}{r_1}\right)^i$$
$$+ (r_2 - k_1k_2 - 1){r_1}^{n-1}\sum_{i=0}^{n-1} i\left(\frac{k_1k_2}{r_1}\right)^i$$
$$+ (2k_1k_2 + 1 - r_1){r_2}^{n-1}\sum_{i=0}^{n-1}\left(\frac{k_1k_2}{r_2}\right)^i$$
$$+ (k_1k_2 + 1 - r_1){r_2}^{n-1}\sum_{i=0}^{n-1} i\left(\frac{k_1k_2}{r_2}\right)^i$$

By using the two formulas 3 and 4, we compute these sums, we get

$$(r_2 - r_1) w_{k_1,k_2,0,\mathrm{n}} = (r_2 - 2k_1k_2 - 1) {r_1}^{n-1} \frac{1 - \left(\frac{k_1k_2}{r_1}\right)^n}{1 - \left(\frac{k_1k_2}{r_1}\right)}$$

$$+ (r_2 - k_1k_2 - 1) {r_1}^{n-1} \frac{n\left(\frac{k_1k_2}{r_2}\right)^{n-1}\left(\frac{k_1k_2}{r_2} - 1\right) + 1 - \left(\frac{k_1k_2}{r_2}\right)^n}{\left(\frac{k_1k_2}{r_2} - 1\right)^2}$$

$$+ (2k_1k_2 + 1 - r_1) {r_2}^{n-1} \frac{1 - \left(\frac{k_1k_2}{r_2}\right)^n}{1 - \frac{k_1k_2}{r_2}}$$

$$+ (k_1k_2 + 1 - r_1) {r_2}^{n-1} \frac{n\left(\frac{k_1k_2}{r_2}\right)^{n-1}\left(\frac{k_1k_2}{r_2} - 1\right) + 1 - \left(\frac{k_1k_2}{r_2}\right)^n}{\left(\frac{k_1k_2}{r_2} - 1\right)^2}$$

$$(r_2 - r_1) w_{k_1,k_2,0,\mathrm{n}} = n(k_1k_2)^n \left(\frac{k_1k_2 + 1 - r_1}{k_1k_2 - r_2} + \frac{r_2 - k_1k_2 - 1}{k_1k_2 - r_1}\right)$$

$$+ (k_1k_2)^n \left(\frac{r_2 - 2k_1k_2 - 1}{k_1k_2 - r_1} + \frac{2k_1k_2 + 1 - r_1}{k_1k_2 - r_2}\right)$$

$$-(k_1k_2)^n \left(\frac{k_1k_2(r_2 - k_1k_2 - 1)}{(k_1k_2 - r_1)^2} + \frac{k_1k_2(k_1k_2 + 1 - r_1)}{(k_1k_2 - r_2)^2}\right)$$

$$+ {r_1}^n \left(-\frac{r_2 - 2k_1k_2 - 1}{k_1k_2 - r_1} + \frac{k_1k_2(r_2 - k_1k_2 - 1)}{(k_1k_2 - r_1)^2}\right)$$

$$+ {r_2}^n \left(-\frac{2k_1k_2 + 1 - r_1}{k_1k_2 - r_2} + \frac{k_1k_2(k_1k_2 + 1 - r_1)}{(k_1k_2 - r_2)^2}\right)$$

Developing this summation we found:

$$\frac{k_1k_2 + 1 - r_1}{k_1k_2 - r_2} + \frac{r_2 - k_1k_2 - 1}{k_1k_2 - r_1} = 0$$

$$\frac{r_2 - 2k_1k_2 - 1}{k_1k_2 - r_1} + \frac{2k_1k_2 + 1 - r_1}{k_1k_2 - r_2} - \frac{k_1k_2(r_2 - k_1k_2 - 1)}{(k_1k_2 - r_1)^2} + \frac{k_1k_2(k_1k_2 + 1 - r_1)}{(k_1k_2 - r_2)^2} = -2$$

$$-\frac{r_2 - 2k_1k_2 - 1}{k_1k_2 - r_1} + \frac{k_1k_2(r_2 - k_1k_2 - 1)}{(k_1k_2 - r_1)^2} = 1$$

$$-\frac{2k_1k_2 + 1 - r_1}{k_1k_2 - r_2} + \frac{k_1k_2(k_1k_2 + 1 - r_1)}{(k_1k_2 - r_2)^2} = 1$$

Hence the result.

4. CONCLUSION

In this work, we aimed to give an efficient way computing the number of spanning trees in the k_1, k_2-multiple-(vertices, edges) wheel graph, providing a new combinatorial method. The advantage of our technique lies in the avoidance of laborious computation of determinant of Laplacian matrix that is needed for a generic method for determining spanning trees.

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