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Residual Power Series Method for Solving Kaup-Boussinesq System

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ABSTRACT

In this paper, a residual power series method (RPSM) is applied to one of the coupled nonlinear system which is called Kaup-Boussinesq system. The approximate solution obtained by RPSM is compared with the exact solution as well as the solution obtained by HPM. The results reveal that the RPSM is convenient, quite accurate to such types of nonlinear partial differential equations, successful and efficient method.

Key words: Residual power series method (RPSM), Homotopy perturbation method (HPM), Kaup-Boussinesq system (KB).

1. INTRODUCTION

A large number of problems in engineering and physics are described by partial differential equations (PDEs) with choosing suitable initial and boundary value conditions. For instance, in electromagnetic theory, traffic flow, fluid dynamics and many other fields the role of Partial differential equations recognized as an important role [13].

In applied mathematics and physics nonlinear phenomena play a vital role. Authors might know the described process deeply by the results of solving nonlinear equations. However, obtaining the exact solution for these problems could be difficult. Moreover, recently, a numerical analysis and exact solution for nonlinear partial equations has faced a huge expansion. For determining a solution, approximate or exact, analytical or numerical, to nonlinear models much attention have been devoted to the search for better and more efficient solution methods. It is also interesting and significant to find the approximate solutions of these nonlinear equations [6]. Many methods have been developed to solve nonlinear partial differential equations such as: Homotopy perturbation method and Residual power series method.

He [8]-[11] is the first who proposed the homotopy perturbation method. The problems can easily be solved by this method because it deforms a difficult problem into simple problems [17]. One of the newly established analytical methods for strongly nonlinear problems based on a series approximation is the HPM. Moreover, it is successfully and efficiently proved that in solving a wide class of nonlinear differential equations (NLDEs) such as: various physics and engineering problems [23], [25]. Also the KB-system has solved numerically by HPM in [24].

Abu Arqub [20] developed the RPSM which is an efficient numerical and analytical method for the determining the coefficients of power series solutions for a class of fuzzy differential equation. Moreover, there is a success in applying the RPSM for getting numerical solutions for many other problems and good examples of this are generalized Lane– Emden equation [19], composite and non-composite fractional differential equations [2] and regular initial value problems [16]. The power series solutions for strongly linear and nonlinear equations are obtained without linearization, discretization or perturbation by this effective and easy method [21], and by chain of linear equations of one or more variable we can compute the coefficient of power series. There are many features of RPS method including [18], [21]: a Taylor expansion of the solution can be obtained by this method which can obtain the exact solution when it is a polynomial. Furthermore, for each arbitrary point in a given interval the solutions and all of its derivatives could be appropriate. Second characteristic is less time, high precision and small computational are required by the RPS method. Many researches have been done by using RPSM such as: [1], [26] and [18].

The Kaup-Boussinesq System is a coupled system of nonlinear partial differential equations and has been derived for an internal wave system , and it has also been derived as a model for surface waves in the context of Boussinesq scaling [3],[12].

The Kaup-Boussinesq system [12]:

$$\left.\begin{aligned} u_t - v_{xxx} - 2(vu)_x &= 0, \\ v_t - u_x - 2vv_x &= 0. \end{aligned}\right\} \tag{1}$$

With the initial conditions:

$$u(x,0) = \frac{w^2}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1+\tanh\left(\frac{wx}{2}\right)\right)^2,$$

And

$$v(x,0) = \frac{-w}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right).$$

Where $u = u(x,t)$ indicate to the height of the water surface above a horizontal bottom, $v = v(x,t)$ is related to the horizontal velocity field and w is constant.
It can be called the Kaup-Boussinesq system because of several reasons such as: it uses Boussinesq scaling in the derivation, and it is studied by Kaup [5]. It has been derived also by L. J. F. Broer [14]. Moreover, it belongs to the family of long-waves models developed by Boussinesq, extended by [4], [22] and many others. Finlay, because it has a coupled set of equations which are both nonlinear the KB-system also known as a complex system.

Recently, a large number of researches have been done on solving KB-system such as: Aminikhah, Sheikhani and Rezazadeh [7] work on travelling wave solution of nonlinear systems of PDEs by using the functional variable method. And [12] work on Solitary-wave solutions to a dual equation of the KB system.

2. DESCRIPTION OF THE METHOD:

2.1 Basic idea of Residual power series method:

To use RPSM for obtaining the approximate solution of nonlinear partial differential equation, we suppose that a general nonlinear partial differential equation [15], [26].

$$D_t u(x,t) = N(u) + R(u), \quad (2)$$

Where $N(u)$ is a nonlinear term and $R(u)$ a linear term.

The initial condition:

$$u(x,0) = f_0(x) \quad (3)$$

The RPSM suggest the solution for (2) as a power series,

$$u_m(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n, \quad x \in I, 0 \le t < \mathbb{R} \quad (4)$$

Next, we let $u_m(x,t)$ denote the m^{th} truncated series of $u(x,t)$,

$$u_m(x,t) = \sum_{n=0}^{m} f_n(x)t^n, \quad (5)$$

The 0^{th} RPS approximation solution of $u(x,t)$ is:

$$u_0(x,t) = u(x,0) = f_0(x) \quad (6)$$

Equation (4) can be written as:

$$u_m(x,t) = f_0(x) + \sum_{n=1}^{m} f_n(x)t^n, \quad m = 1,2,3,.... \quad (7)$$

We define the residual function for (2) as:

$$Res_u(x,t) = D_t(x,t) - N(u) - R(u). \quad (8)$$

Therefore, the m^{th} residual function $Res_{u,m}$ is of the form:

$$Res_{u,m}(x,t) = D_t u_m(x,t) - N(u_m) - R(u_m). \quad (9)$$

We state some results of $Res_u(x,t)$ from [19], [20], [21], which are essential in RPSM:

i. $Res(x,t) = 0$

ii. $\lim_{m\to\infty} Res_m(x,t) = Res(x,t)$ for all $x \in I$ and $t \ge 0$ (10)

iii. $D_t^r Res_m(x,t) = 0, \quad r = 0,1,2,4,\dots,m.$

Then, we should find coefficients $f_1(x), f_2(x), ...$ of the residual power series solution (7) as follows:

Substitute the m^{th} truncated series into the equation (9) and calculate the derivative D_t^{m-1} of $Res_m(x,t), m = 1,2,3,...$ together with equation (10), the following algebraic system is obtained:

$$D_t^{m-1} Res_{u,m}(x,0) = 0, \quad m = 1,2,3,.... \quad (11)$$

3. NUMERICAL APPLICATIONS:

We applied the presented method for solving the following example:

$$u_t - v_{xxx} - 2(vu)_x = 0,$$
$$v_t - u_x - 2vv_x = 0.$$

With the initial conditions:

$$u(x,0) = \frac{w^2}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1 + \tanh\left(\frac{wx}{2}\right)\right)^2,$$

And

$$v(x,0) = \frac{-w}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right).$$

Where $w = 1.5$, and with the soliton solutions [7]:

$$u(x,t) = \frac{w^2}{2}\left(1 + \tanh\left(\frac{w(x-wt)}{2}\right)\right) - \frac{w^2}{4}\left(1 + \tanh\left(\frac{w(x-wt)}{2}\right)\right)^2$$

And

$$v(x,t) = \frac{-w}{2}\left(1 + \tanh\left(\frac{w(x-wt)}{2}\right)\right).$$

3.1 The solution of the Kaup-Boussinesq system by RPSM:

We suppose that the KB-equations (1), subject to the initial conditions:

$$u(x,0) = f_0(x), \quad v(x,0) = g_0(x). \quad (12)$$

Constructing a power series solution to the system (1) by its power series expansion among its truncated residual function is the main purpose.

The following summarized steps are the procedure of the RPSM for system (1) and (12):

1^{st} step. Suggest that the solution to the system (1) and (12) as a power series can be written as:

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)t^n, \quad v(x,t) = \sum_{n=0}^{\infty} g_n(x)t^n, \quad (13)$$

Next, we can represent the m[th] truncated series of $u(x,t), v(x,t)$ as:

$$u_m(x,t) = \sum_{n=0}^{m} f_n(x)t^n, \quad v_m(x,t) = \sum_{n=0}^{m} g_n(x)t^n, \tag{14}$$

If we take $m = 0$ by the initial conditions (12), it becomes easy to verify that the zeros RPS truncated solutions of $u(x,t)$ and $v(x,t)$ are:

$$u_0(x,t) = f_0(x) = u(x,0), \quad v_0(x,t) = g_0(x) = v(x,0), \tag{15}$$

Therefore, the m[th] truncated series of $u(x,t), v(x,t)$ can be rewritten as:

$$u_m(x,t) = f_0(x) + \sum_{n=1}^{m} f_n(x)t^n, \quad v_m(x,t) = g_0(x) + \sum_{n=1}^{m} g_n(x)t^n, \tag{16}$$

when $x \in I, 0 \le t < \mathbb{R}$.

By representation of $u_m(x,t)$, $v_m(x,t)$, and after $f_i(x)$ and $g_i(x), i = 1,2,\dots,m$ are available the m[th] RPS approximate solution will be obtained.

2[nd] step. The residual functions for system (1) and (12) are defined respectively:

$$Res_u(x,t) = D_t u - \frac{\partial^3 v}{\partial x^3} - 2v\frac{\partial u}{\partial x} - 2u\frac{\partial v}{\partial x}, \quad Res_v(x,t) = D_t v - \frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \tag{17}$$

Furthermore, the m[th] residual functions take the forms:

$$Res_{u,m}(x,t) = D_t u_m - \frac{\partial^3 v_m}{\partial x^3} - 2v_m\frac{\partial u_m}{\partial x} - 2u_m\frac{\partial v_m}{\partial x}, \quad Res_{v,m}(x,t) = D_t v_m - \frac{\partial u_m}{\partial x} - 2v_m\frac{\partial v_m}{\partial x} \tag{18}$$

As (11) we have the following algebraic system:

$$D_t^{m-1} Res_{u,m}(x,0) = 0$$

$$D_t^{m-1} Res_{v,m}(x,0) = 0, \quad m = 1,2,3,\dots.$$

3[rd] step. After the above algebraic system have solved, we will have $f_i(x)$ and $g_i(x), i = 0,1,2,\dots,m$. That's why, the m[th] RPS approximation is obtained.

4[th] step. The first approximate solution will deduce in details. For $m = 1$, the first RPS approximate solution can be written as:

$$u_1(x,t) = f_0(x) + f_1(x)t, \quad v_1(x,t) = g_0(x) + g_1(x)t, \tag{19}$$

By substituting equations (19) into (18) respectively, we get the first residual functions:

$$D_t^{1-1} Res_{u,1}(x,t) = D_t^{1-1}\left[D_t u_1 - \frac{\partial^3 v_1}{\partial x^3} - 2v_1\frac{\partial u_1}{\partial x} - 2u_1\frac{\partial v_1}{\partial x}\right]_{t=0} = 0$$

$$Res_{u,1}(x,t) = \left[D_t u_1 - \frac{\partial^3 v_1}{\partial x^3} - 2v_1\frac{\partial u_1}{\partial x} - 2u_1\frac{\partial v_1}{\partial x}\right]_{t=0} = 0$$

$$= [\tfrac{\partial}{\partial t}(f_0(x) + f_1(x)t) - (g_0'''(x) + g_1'''(x)t) - 2(g_0(x) + g_1(x)t)(f_0'(x) + f_1'(x)t) - 2(f_0(x) + f_1(x)t)(g_0'(x) + g_1'(x)t)]_{t=0} = 0$$

$$= [f_1(x) - g_0'''(x) - g_1'''(x)t - 2g_0(x)f_0'(x) - 2g_0(x)f_1'(x)t - 2g_1(x)f_0'(x)t - 2g_1(x)f_1'(x)t^2 - 2f_0(x)g_0'(x) - 2f_0(x)g_1'(x)t - 2f_1(x)g_0'(x)t - 2f_1(x)g_1'(x)t^2]_{t=0} = 0$$

$$= f_1(x) - g_0'''(x) - 2g_0(x)f_0'(x) - 2f_0(x)g_0'(x) = 0$$

And

$$D_t^{1-1} Res_{v,1}(x,t) = D_t^{1-1}\left[D_t v_1 - \frac{\partial u_1}{\partial x} - 2v_1\frac{\partial v_1}{\partial x}\right]_{t=0} = 0$$

$$Res_{v,1}(x,t) = \left[D_t v_1 - \frac{\partial u_1}{\partial x} - 2v_1\frac{\partial v_1}{\partial x}\right]_{t=0} = 0$$

$$= \left[\tfrac{\partial}{\partial t}(g_0(x) + g_1(x)t) - (f_0'(x) + f_1'(x)t) - 2(g_0(x) + g_1(x)t)(g_0'(x) + g_1'(x)t)\right]_{t=0} = 0$$

$$= g_1(x) - f_0'(x) - f_1'(x)t - 2g_0(x)g_0'(x) - 2g_0(x)g_1'(x)t - 2g_1(x)g_0'(x)t - 2g_1(x)g_1'(x)t^2]_{t=0} = 0$$

$$= g_1(x) - f_0'(x) - 2g_0(x)g_0'(x) = 0$$

According to $Res_{u,1}(x,0) = Res_{v,1}(x,0) = 0$, we get the following algebraic system:

$$f_1(x) - g_0'''(x) - 2g_0(x)f_0'(x) - 2f_0(x)g_0'(x) = 0,$$

$$g_1(x) - f_0'(x) - 2g_0(x)g_0'(x) = 0,$$

Therefore,

$$f_1(x) = g_0'''(x) + 2g_0(x)f_0'(x) + 2f_0(x)g_0'(x), \quad g_1(x) = f_0'(x) + 2g_0(x)g_0'(x), \tag{20}$$

Then by equation (19), we get:

$$u_1(x,t) = f_0(x) + \left(f_1(x) - g_0'''(x) - 2g_0(x)f_0'(x) - 2f_0(x)g_0'(x)\right)t$$

$$v_1(x,t) = g_0(x) + \left(f_0'(x) + 2g_0(x)g_0'(x)\right)t$$

The higher degree of approximate solution can be obtained in the same way, when $m = 2, 3, \dots$.

3.2 Applying RPSM for solving KB-system:

For $m = 0$, we get:

$$u_0(x,t) = f_0(x) = \frac{w^2}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1+\tanh\left(\frac{wx}{2}\right)\right)^2$$

$$v_0(x,t) = g_0(x) = \frac{-w}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right)$$

For $m = 1$, we get

$$u_1(x,t) = f_0(x) + \left(f_1(x) - g_0'''(x) - 2g_0(x)f_0'(x) - 2f_0(x)g_0'(x)\right)t$$

$$u_1(x,t) = \frac{w^2}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1+\tanh\left(\frac{wx}{2}\right)\right)^2 + \frac{w^4}{8}\operatorname{sech}^4\left(\frac{wx}{2}\right)t + \frac{w^4}{4}\tanh\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t - \frac{w^4}{8}\operatorname{sech}^2\left(\frac{wx}{2}\right)t + \frac{w^4}{8}\tanh^2\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t.$$

And

$$v_1(x,t) = g_0(x) + \left(f_0'(x) + 2g_0(x)g_0'(x)\right)t.$$

$$v_1(x,t) = \frac{-w}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right) + \frac{w^3}{4}\operatorname{sech}^2\left(\frac{wx}{2}\right)t.$$

Then by the same way find $u_2(x,t)$, $v_2(x,t)$ and so on. The results obtained in this paper can be compared with [24], which is about solving KB-system numerically by HPM and HAM.

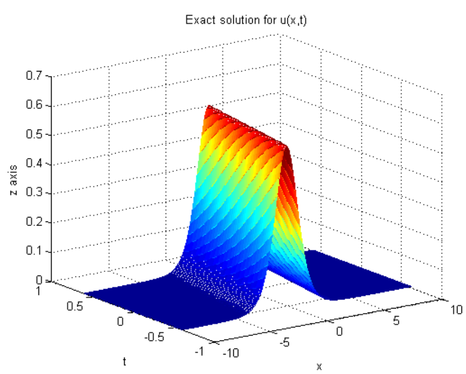

Figure 1: Exact solution for $u(x,t)$.

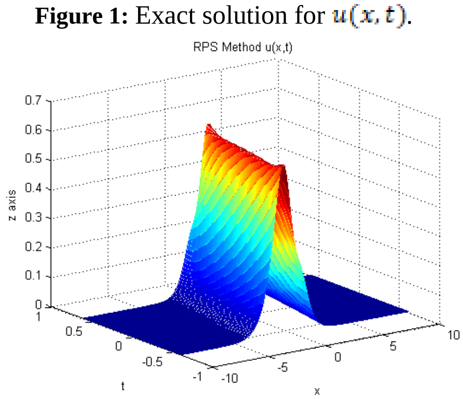

Figure 2: The solution for $u(x,t)$ by RPSM.

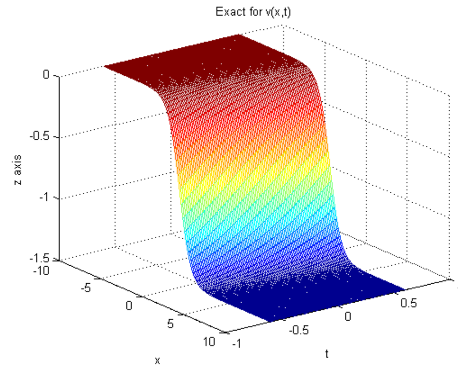

Figure 3: Exact solution for $v(x,t)$.

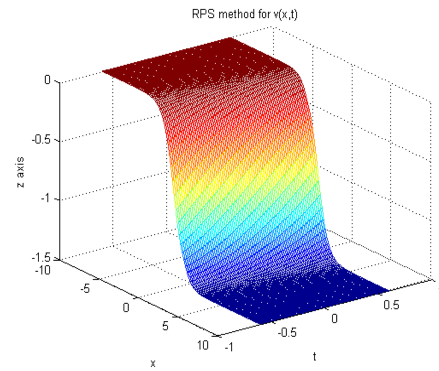

Figure 4: The solution for $v(x,t)$ by RPSM.

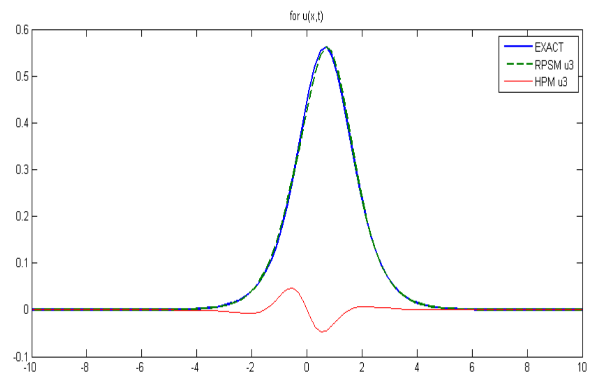

Figure 5: The curves of Exact solution, RPSM and HPM, while selecting $u_3(x,t)$ from both methods, when $x \in [-10,10]$, $t = 0.44$ and $w = 1.5$.

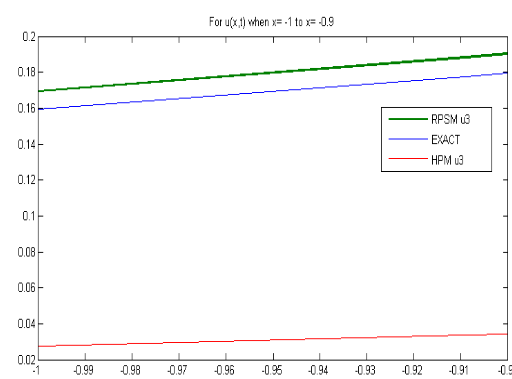

Figure 6: Zooming curves of Exact solution, RPSM and HPM, while selecting $u_3(x,t)$ from both methods, when $x \in [-1,-0.9]$, $t = 0.44$ and $w = 1.5$.

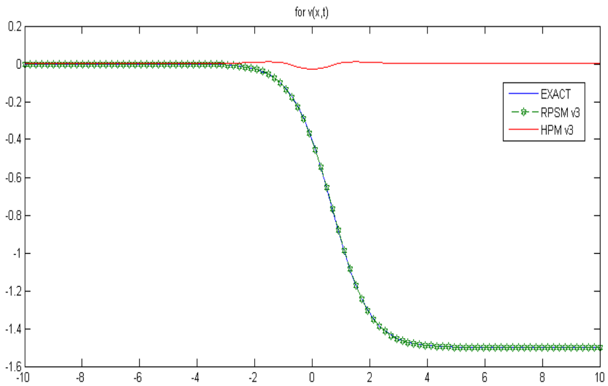

Figure 7: The curves of Exact solution, RPSM and HPM, while selecting $v_3(x,t)$ from both methods, when $x \in [-10,10]$, $t = 0.44$ and $w = 1.5$.

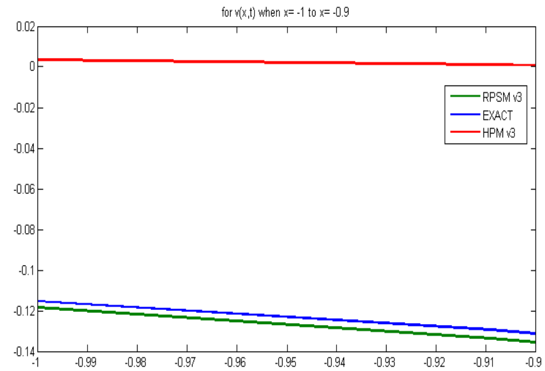

Figure 8: Zooming curves of Exact solution, RPSM and HPM, while selecting $v_3(x,t)$ from both methods, when $x \in [-1,-0.9]$, $t = 0.44$ and $w = 1.5$.

Table 1: Absolute errors $|u_{exact} - u_3|$ for RPSM and HPM, when $x \in [-10,10]$ and $t = 0.44$.

x	t	EXACT	RPSM(u_3)	HPM(u_3)	|EXACT-RPSM(u_3)|	|EXACT-HPM(u_3)|
-4.5	0.44	9.087E-04	8.276E-04	-3.975E-04	8.108E-05	1.306E-03
-4.3		1.230E-03	1.121E-03	-5.349E-04	1.086E-04	1.765E-03
-4.1		1.665E-03	1.520E-03	-7.181E-04	1.449E-04	2.383E-03
-3.9		2.253E-03	2.060E-03	-9.612E-04	1.923E-04	3.214E-03
-3.7		3.048E-03	2.794E-03	-1.281E-03	2.534E-04	4.329E-03
-3.5		4.123E-03	3.792E-03	-1.698E-03	3.304E-04	5.821E-03
-1.3		1.049E-01	1.113E-01	8.679E-03	6.360E-03	9.622E-02
-1.1		1.372E-01	1.463E-01	2.035E-02	9.125E-03	1.169E-01
-0.9		1.775E-01	1.881E-01	3.364E-02	1.061E-02	1.439E-01
-0.7		2.263E-01	2.353E-01	4.436E-02	8.976E-03	1.819E-01
-0.5		2.832E-01	2.860E-01	4.673E-02	2.861E-03	2.365E-01
-0.3		3.463E-01	3.393E-01	3.632E-02	6.968E-03	3.100E-01
5.2		2.676E-03	2.627E-03	1.619E-04	4.930E-05	2.515E-03
5.4		1.978E-03	1.941E-03	1.198E-04	3.658E-05	1.858E-03
5.6		1.462E-03	1.434E-03	8.857E-05	2.711E-05	1.373E-03
5.8		1.080E-03	1.060E-03	6.548E-05	2.008E-05	1.014E-03
6.0		7.977E-04	7.829E-04	4.840E-05	1.486E-05	7.493E-04
6.2		5.893E-04	5.783E-04	3.577E-05	1.099E-05	5.535E-04
Mean Square error					1.944E-05	2.903E-02

Table 2: Absolute errors $|v_{exact} - v_3|$ for RPSM and HPM, when $x \in [-10,10]$ and $t = 0.44$.

x	t	EXACT	RPSM(v_3)	HPM(v_3)	|EXACT-RPSM(v_3)|	|EXACT-HPM(v_3)|
-4.5	0.44	-6.060E-04	-5.512E-04	2.674E-04	5.487E-05	8.734E-04
-4.3		-8.204E-04	-7.465E-04	3.609E-04	7.390E-05	1.181E-03
-4.1		-1.111E-03	-1.011E-03	4.866E-04	9.935E-05	1.597E-03
-3.9		-1.503E-03	-1.370E-03	6.550E-04	1.332E-04	2.158E-03
-3.7		-2.035E-03	-1.857E-03	8.801E-04	1.780E-04	2.915E-03
-3.5		-2.754E-03	-2.517E-03	1.179E-03	2.367E-04	3.933E-03
-1.3		-7.354E-02	-7.409E-02	9.416E-03	5.548E-04	8.295E-02
-1.1		-9.787E-02	-1.000E-01	6.532E-03	2.127E-03	1.044E-01
-0.9		-1.295E-01	-1.337E-01	1.077E-03	4.157E-03	1.306E-01
-0.7		-1.702E-01	-1.764E-01	-6.891E-03	6.204E-03	1.633E-01
-0.5		-2.215E-01	-2.290E-01	-1.628E-02	7.477E-03	2.052E-01
-0.3		-2.850E-01	-2.921E-01	-2.490E-02	7.102E-03	2.601E-01
5.2		-1.498E+00	-1.498E+00	1.083E-04	3.316E-05	1.498E+00
5.4		-1.499E+00	-1.499E+00	8.005E-05	2.455E-05	1.499E+00
5.6		-1.499E+00	-1.499E+00	5.916E-05	1.817E-05	1.499E+00
5.8		-1.499E+00	-1.499E+00	4.372E-05	1.343E-05	1.499E+00
6.0		-1.499E+00	-1.499E+00	3.230E-05	9.931E-06	1.500E+00
6.2		-1.500E+00	-1.500E+00	2.386E-05	7.340E-06	1.500E+00
Mean Square error					3.906E-06	9.776E-01

4. CONCLUSION

In this paper, the RPSM is applied to find the approximate solution for nonlinear Kaup-Boussinesq system, and compared it with the results obtained by HPM while choosing u_3 and v_3 from each method. The results shows that RPSM is a very accurate, effective and powerful method for solving the presented nonlinear system, and were closer to the exact solution than HPM while selecting the third approximation u_3 andv_3 as shown in figures (6, 8) and tables (1, 2).

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