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Periodic solutions for nonlinear systems of integro-differential equations of Volterra- Friedholm type

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ABSTRACT

In this work, we study the periodic solutions for nonlinear systems of integro-differential equations of Volterra- Friedholm. The numerical analytic method has been used by (Samoilenko A. M.) to investigate the existence and approximation of periodic solutions for certain of nonlinear systems of integro-differential equations. Also these methods could be developed and extended throughout the study.

Key words: Numerical-analytic method, periodic integro-differential equations of Volterra-Friedholm, stability solutions.

1. INTRODUCTION

In recent years, Samoilenko assumes that the numerical analytic method to study the periodic solutions for ordinary differential equations. In the original works of Samoilenk and Ronto [16,17] the approach used and described here had been referred to as the numerical-analytic based upon successive approximations. The idea of the method, originally aimed at the investigation of periodic solution only, had been later applied in studies [2,3,8,9,10,11,13,14,15].
Also, it should be noted that appropriate versions of the method considered can be applied in many situations for handling periodic in the cases of the systems of first or second order ordinary differential equations, integro-differential equations, equations with retarded arguments, systems containing unknown parameters, and countable systems of differential equations. A survey of the investigations on the subject can be found in the studies and researches [4,5,6,12,13,18,19,20].
Butris[1], assumes the both methods Picard approximation and Banach fixed point theorem to study the existence, uniqueness and stability solution of Volterra- Friedholm of integro-differential equations which has the following form:-

$$\frac{dx}{dt} = Ax + By + \int_{-\infty}^{t} K(t,s) f\big(s, x(s), y(s)\big) ds + \int_a^b G(t,s) g\big(s, x(s), y(s)\big) ds \qquad \text{(VF1)}$$

$$\frac{dy}{dt} = Cx + Ey + \int_{-\infty}^{t} \varphi\big(t, s, x(s), y(s)\big) ds + \int_a^b \psi\big(t, s, x(s), y(s)\big) ds \qquad \text{(VF2)}$$

where $x \in D \subseteq R^n$ and $y \in D_1 \subseteq R^n$, D and D_1 are closed and bounded domains.

In this work, we study the periodic solutions for nonlinear systems of integro-differential equations of Volterra- Friedholm (VF1) and (VF2).

Let the vector functions $f(t,x,y)$, $g(t,x,y)$, $\varphi(t,s,x,y)$ and $\psi(t,s,x,y)$ defined on the domain

$$\left.\begin{array}{l} (t,s,x) \in R^1 \times R^1 \times D = \\ (-\infty,\infty) \times (-\infty,\infty) \times D \\ (t,s,y) \in R^1 \times R^1 \times D_1 = \\ (-\infty,\infty) \times (-\infty,\infty) \times D_1 \end{array}\right\} \qquad (1)$$

and periodic in t of period T. Assume that the vector functions $f(t,x,y)$, $g(t,x,y)$, $\varphi(t,s,x,y)$ and $\psi(t,s,x,y)$ are satisfying the following inequalities:

$$\|f(t,x,y)\| \le M_1, \|g(t,x,y)\| \le M_2 \qquad (2)$$

$$\|\varphi(t,s,x,y)\| \le N_1 \|H(t,s)\|, \ \|\psi(t,x,y)\| \le N_2 \qquad (3)$$

$$\|f(t,x_1,y_1) - f(t,x_2,y_2)\| \le V_1\|x_1 - x_2\| + V_2\|y_1 - y_2\| \qquad (4)$$

$$\|g(t,x_1,y_1) - g(t,x_1,y_1)\| \le U_1\|x_1 - x_2\| + U_2\|y_1 - y_2\| \qquad (5)$$

$$\|\varphi(t,s,x_1,y_1) - \varphi(t,s,x_2,y_2)\| \le \|H(t,s)\|(L_1\|x_1 - x_2\| + L_2\|y_1 - y_2\|) \qquad (6)$$

$$\|\psi(t,s,x_1,y_1) - \psi(t,s,x_2,y_2)\| \le \|F(t,s)\|(J_1\|x_1 - x_2\| + J_2\|y_1 - y_2\|), \qquad (7)$$

for all $t \in R^1, s \in R^1$ and $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$ where $M_1, M_2, G_1, N_1, N_2, L_1, L_2, J_1$ and J_2 are positive constants.

Let each of the functions $K(t,s)$ and $G(t,s)$ be a kernel for the equations (VF1) and (VF2) provided that:-

$$\|K(t,s)\| \le \delta_1 e^{-\lambda_1(t-s)}, \|H(t,s)\| \le \delta_2 e^{-\lambda_2(t-s)} \qquad (8)$$

$$\|G(t,s)\| \le G_1, \|F(t,s)\| \le F_1 \qquad (9)$$

We define the non-empty sets as:

$$\left.\begin{array}{l} D_f = D - P_1\|x_0\| + \beta_1(t) A_2[B\|y_0\| + \Lambda_1] \\ D_{1f} = D_1 - P_2\|y_0\| + \beta_2(t) E_2[C\|x_0\| + \Lambda_2] \end{array}\right\} \qquad (10)$$

where

$$\beta_1(t) = \frac{t\left(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|I\|\right) + T\left(e^{\|A\|T} - e^{\|A\|(T-t)}\right)}{e^{\|A\|T} - \|I\|}, \quad \beta_2(t) = \frac{t\left(2e^{\|E\|(T-t)} - e^{\|E\|T} - \|I\|\right) + T\left(e^{\|E\|T} - e^{\|E\|(T-t)}\right)}{e^{\|E\|T} - \|I\|},$$

$P_1 = \|e^{At} - At - I\|$, $P_2 = \|e^{Et} - Et - I\|$, $A_2 = \left\|e^{A(t-s)}\right\|$, $E_2 = \left\|e^{E(t-s)}\right\|$, $\Lambda_1 = \frac{\delta_1}{\lambda_1} M_1 + (b-a) G_1 M_2$, $\Lambda_2 = \frac{\delta_2}{\lambda_2} N_1 + (b-a) N_2$, $\|.\| = \max_{t \in [0,T]} |.|$, and I is identity matrix .

As well as, we suppose that the maximum value of the following matrix is:

$Q_0=\begin{pmatrix} A_2\frac{T}{2}\Omega_1 & A_2\frac{T}{2}\Omega_2 \\ E_2\frac{T}{2}\Omega_3 & E_2\frac{T}{2}\Omega_4 \end{pmatrix}$, less than one i. e.

$$\lambda_{max}(Q_0)<1 \tag{11}$$

where $\Omega_1=\frac{\delta_1}{\lambda_1}V_1+(b-a)G_1U_1$, $\Omega_2=B+\frac{\delta_1}{\lambda_1}V_2+(b-a)G_1U_2$, $\Omega_3=C+\frac{\delta_2}{\lambda_2}L_1+(b-a)F_1J_1$ and $\Omega_4=\frac{\delta_2}{\lambda_2}L_2+(b-a)F_1J_2$.

Define a sequence of functions $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ by:

$$x_{m+1}(t,x_0,y_0)=(e^{At}-At)x_0+\int_0^t\Big(e^{A(t-s)}\Big(By_m(s,x_0,y_0)+\int_{-\infty}^t K(t,s)f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds+\int_a^b G(t,s)g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds\Big)-\frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\Big(By_m(s,x_0,y_0)+\int_{-\infty}^T K(T,s)f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds+\int_a^b G(T,s)g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds\Big)ds\Big)ds, \tag{12}$$

with $x_0(0,x_0,y_0)=x_0$, $m=0,1,2,\dots\dots$,

and

$$y_{m+1}(t,x_0,y_0)=(e^{Et}-Et)y_0+\int_0^t\Big(e^{E(t-s)}(Cx_m(s,x_0,y_0)+\int_{-\infty}^t\varphi(t,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds+\int_a^b\psi(t,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds-\frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\Big(Cx_m(s,x_0,y_0)+\int_{-\infty}^T\varphi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds+\int_a^b\psi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds\Big)ds\Big)ds, \tag{13}$$

with $y_0(0,x_0,y_0)=y_0$, $m=0,1,2,\dots\dots$

Lemma1. Let the vector function $f(t,x)$ is defined and continuous on the interval $[0,T]$, then the following inequality $\left\|\int_0^t\Big(f(s,x(s))-\frac{1}{T}\int_0^T f(s,x(s))ds\Big)ds\right\|\le\alpha(t)M$

is holds, where $\alpha(t)=2t\left(1-\frac{t}{T}\right)$ and $M=\max_{t\in[0,T]}|f(t,x)|$.(For the proof see[17]).

By using lemma1, we can formulate the following lemma.

Lemma2. Suppose that the vector functions $f(t,x,y)$, $g(t,x,y)$, $\varphi(t,s,x,y)$ and $\psi(t,s,x,y)$ are defined, continuous and periodic in t of period T on the domain (1). Then the following inequality holds:

$$\begin{pmatrix}\|R_1(t,x_0,y_0)\|\\ \|R_2(t,x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}\beta_1(t)A_2(B\|y_0\|+\Lambda_1)\\ \beta_2(t)E_2(C\|x_0\|+\Lambda_2)\end{pmatrix} \tag{14}$$

where

$$R_1(t,x_0,y_0)=\int_0^t\Big(e^{A(t-s)}\Big(By_0+\int_{-\infty}^t K(t,s)f(s,x_0,y_0)ds+\int_a^b G(t,s)g(s,x_0,y_0)ds\Big)-\frac{A}{e^{AT-I}}\int_0^T e^{A(T-s)}\Big(By_0+\int_{-\infty}^T K(T,s)f(s,x_0,y_0)ds+\int_a^b G(T,s)g(s,x_0,y_0)ds\Big)ds\Big)ds$$

$$R_2(t,x_0,y_0)=\int_0^t\Big(e^{E(t-s)}\Big(Cx_0+\int_{-\infty}^t\varphi(t,s,x_0,y_0)ds+\int_a^b\psi(t,s,x_0,y_0)ds\Big)-\frac{E}{e^{ET-I}}\int_0^T e^{E(T-s)}\Big(Cx_0+\int_{-\infty}^T\varphi(T,s,x_0,y_0)ds+\int_a^b\psi(T,s,x_0,y_0)ds\Big)ds\Big)ds$$, and

$$\beta_1(t)\le\frac{t\big(2e^{\|A\|(T-t)}-e^{\|A\|T}-\|I\|\big)+T\big(e^{\|A\|T}-e^{\|A\|(T-t)}\big)}{e^{\|A\|T}-\|I\|}$$

$$\beta_2(t)\le\frac{t\big(2e^{\|E\|(T-t)}-e^{\|E\|T}-\|I\|\big)+T\big(e^{\|E\|T}-e^{\|E\|(T-t)}\big)}{e^{\|E\|T}-\|I\|}$$

for $0\le t\le T$, $\beta_1(t)\le\frac{T}{2}$, $\beta_2(t)\le\frac{T}{2}$

Proof. Since

$$\begin{pmatrix}\|R_1(t,x_0,y_0)\|\\ \|R_2(t,x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}\Big[\|I\|-\Big(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|I\|}\Big)\Big]\int_0^t\|e^{A(t-s)}\|\begin{pmatrix}B\|y_0\|+\int_{-\infty}^t\|K(t,s)\|\|f(s,x_0,y_0)\|ds\\ +\int_a^b\|G(t,s)\|\|g(s,x_0,y_0)\|ds\end{pmatrix}ds\\ +\Big[\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|I\|}\Big]\int_t^T\|e^{A(T-s)}\|\begin{pmatrix}B\|y_0\|+\int_{-\infty}^T\|K(T,s)\|\\ \|f(s,x_0,y_0)\|ds+\int_a^b\|G(T,s)\|\|g(s,x_0,y_0)\|ds\end{pmatrix}ds\\ \Big[\|I\|-\Big(\frac{e^{\|E\|T}-e^{\|E\|(T-t)}}{e^{\|E\|T}-\|I\|}\Big)\Big]\int_0^t\|e^{E(t-s)}\|\begin{pmatrix}C\|x_0\|+\int_{-\infty}^t\|\varphi(t,s,x_0,y_0)\|ds\\ +\int_a^b\|\psi(t,s,x_0,y_0)\|ds\end{pmatrix}ds\\ +\Big[\frac{e^{\|E\|T}-e^{\|E\|(T-t)}}{e^{\|E\|T}-\|I\|}\Big]\int_t^T\|e^{E(T-s)}\|\begin{pmatrix}C\|x_0\|+\int_{-\infty}^T\|\varphi(T,s,x_0,y_0)\|ds\\ +\int_a^b\|\psi(T,s,x_0,y_0)\|ds\end{pmatrix}ds\end{pmatrix}$$

$$\begin{pmatrix}\|R_1(t,x_0,y_0)\|\\ \|R_2(t,x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}\frac{t\big(2e^{\|A\|(T-t)}-e^{\|A\|T}-\|I\|\big)+T\big(e^{\|A\|T}-e^{\|A\|(T-t)}\big)}{e^{\|A\|T}-\|I\|}A_2\\ \Big[B\|y_0\|+\frac{\delta_1}{\lambda_1}M_1+(b-a)G_1M_2\Big]\\ \frac{t\big(2e^{\|E\|(T-t)}-e^{\|E\|T}-\|I\|\big)+T\big(e^{\|E\|T}-e^{\|E\|(T-t)}\big)}{e^{\|E\|T}-\|I\|}E_2\\ \Big[C\|x_0\|+\frac{\delta_2}{\lambda_2}N_1+(b-a)N_2\Big]\end{pmatrix}$$

then

$$\begin{pmatrix}\|R_1(t,x_0,y_0)\|\\ \|R_2(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}\beta_1(t)A_2[B\|y_0\|+\Lambda_1]\\ \beta_2(t)E_2[C\|x_0\|+\Lambda_2]\end{pmatrix}.\blacksquare$$

2.APPROXIMATION OF PERIODIC SOLUTION OF (VF1) AND (VF2).

In this section, we prove the approximation of periodic solution of (VF1) and (VF2) be introduced in the following theorem:-

Theorem3. Let the vector functions $f(t,x,y)$, $g(t,x,y), \varphi(t,s,x,y)$ and $\psi(t,s,x,y)$ are defined, continuous and periodic in t of period T on the domain (1). Suppose that these functions are satisfied the inequalities (2) to (9). Then there exist a sequence of functions (12) and (13) are periodic in t of period T, converges uniformly as $m \to \infty$ in the domain

$$(t,x_0,y_0) \in [0,T]\times D_f \times D_{1f}, \qquad (15)$$

to the limit functions which has the forms:

$$x(t,x_0,y_0) = (e^{At}-At)x_0 + \int_0^t \Big(e^{A(t-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^t K(t,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(t,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^T K(T,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(T,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds, \quad (16)$$

and

$$y(t,x_0,y_0) = (e^{Et}-Et)y_0 + \int_0^t \Big(e^{E(t-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^t \varphi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^T \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds, \quad (17)$$

Provided that:

$$\left.\begin{aligned}&\begin{pmatrix}\|x(t,x_0,y_0)-x_0\|\\ \|y(t,x_0,y_0)-y_0\|\end{pmatrix} \le \begin{pmatrix}P_1\|x_0\|+\beta_1(t)A_2[B\|y_0\|+\Lambda_1]\\ P_2\|y_0\|+\beta_2(t)E_2[C\|x_0\|+\Lambda_2]\end{pmatrix}\\ &\begin{pmatrix}\|x(t,x_0,y_0)-x_m(t,x_0,y_0)\|\\ \|y(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le \begin{pmatrix}Q^m(E-Q)^{-1}(P_1\|x_0\|+\beta_1(t)A_2[B\|y_0\|+\Lambda_1])\\ Q^m(E-Q)^{-1}(P_2\|y_0\|+\beta_2(t)E_2[C\|x_0\|+\Lambda_2])\end{pmatrix}\end{aligned}\right\} \quad (18)$$

Proof. By the lemma2 and using the sequence of functions (12) and (13) when $m=0$, we get

$$\begin{pmatrix}\|x_1(t,x_0,y_0)-x_0\|\\ \|y_1(t,x_0,y_0)-y_0\|\end{pmatrix} \le$$

$$\begin{pmatrix}\|e^{At}-At-I\|\|x_0\| + \Big[\|I\| - \Big(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|I\|}\Big)\Big] \int_0^t \|e^{A(t-s)}\| \begin{pmatrix}B\|y_0\| + \int_{-\infty}^t \|K(t,s)\|\|f(s,x_0,y_0)\|\\ ds + \int_a^b \|G(t,s)\|\|g(s,x_0,y_0)\|ds\end{pmatrix} ds + \Big[\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|I\|}\Big]\int_t^T \|e^{A(T-s)}\| \begin{pmatrix}B\|y_0\| + \int_{-\infty}^T \|K(T,s)\|\|f(s,x_0,y_0)\|ds +\\ \int_a^b \|G(T,s)\|\|g(s,x_0,y_0)\|ds\end{pmatrix} ds\\ \|e^{Et}-Et-I\|\|y_0\| + \Big[\|I\| - \Big(\frac{e^{\|E\|T}-e^{\|E\|(T-t)}}{e^{\|E\|T}-\|I\|}\Big)\Big] \int_0^t \|e^{E(t-s)}\| \begin{pmatrix}C\|x_0\| + \int_{-\infty}^t \|\varphi(t,s,x_0,y_0)\|ds\\ + \int_a^b \|\psi(t,s,x_0,y_0)\|ds\end{pmatrix} ds + \Big[\frac{e^{\|E\|T}-e^{\|E\|(T-t)}}{e^{\|E\|T}-\|I\|}\Big]\int_t^T \|e^{E(T-s)}\| \begin{pmatrix}C\|x_0\| + \int_{-\infty}^T \|\varphi(T,s,x_0,y_0)\|ds +\\ \int_a^b \|\psi(T,s,x_0,y_0)\|ds\end{pmatrix} ds\end{pmatrix}$$

$$\begin{pmatrix}\|x_1(t,x_0,y_0)-x_0\|\\ \|y_1(t,x_0,y_0)-y_0\|\end{pmatrix} \le \begin{pmatrix}P_1\|x_0\| + \frac{t(2e^{\|A\|(T-t)}-e^{\|A\|T}-\|I\|)+T(e^{\|A\|T}-e^{\|A\|(T-t)})}{e^{\|A\|T}-\|I\|}A_2 \Big[B\|y_0\| + \frac{\delta_1}{\lambda_1}M_1 + (b-a)G_1M_2\Big]\\ P_2\|y_0\| + \frac{t(2e^{\|E\|(T-t)}-e^{\|E\|T}-\|I\|)+T(e^{\|E\|T}-e^{\|E\|(T-t)})}{e^{\|E\|T}-\|I\|}E_2 \Big[C\|x_0\| + \frac{\delta_2}{\lambda_2}N_1 + (b-a)N_2\Big]\end{pmatrix}$$

$$\le \begin{pmatrix}P_1\|x_0\|+\beta_1(t)A_2[B\|y_0\|+\Lambda_1]\\ P_2\|y_0\|+\beta_2(t)E_2[C\|x_0\|+\Lambda_2]\end{pmatrix}$$

i. e. $x_1(t,x_0,y_0) \in D$, $y_1(t,x_0,y_0) \in D_1$, for all $t \in [0,T]$, $x_0 \in D_f$ and $y_0 \in D_{1f}$.

Then, by mathematical induction we can prove that:-

$$\begin{pmatrix}\|x_m(t,x_0,y_0)-x_0\|\\ \|y_m(t,x_0,y_0)-y_0\|\end{pmatrix} \le \begin{pmatrix}P_1\|x_0\|+\beta_1(t)A_2[B\|y_{m-1}(t,x_0)\|+\Lambda_1]\\ P_2\|y_0\|+\beta_2(t)E_2[C\|x_{m-1}(t,x_0)\|+\Lambda_2]\end{pmatrix}$$

where $x_m(t,x_0,y_0) \in D$ and $y_m(t,x_0,y_0) \in D_1$ when $x_0 \in D_f$ and $y_0 \in D_{1f}$.

Next, we prove that the sequence of functions (12) and (13) convergent uniformly on the domain (1).

By mathematical induction, we get

$$\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le$$
$$\begin{pmatrix} A_2\beta_1(t)\begin{pmatrix}\left(\frac{\delta_1}{\lambda_1}V_1+(b-a)G_1U_1\right) \\ \|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ +\left(B+\frac{\delta_1}{\lambda_1}V_2+(b-a)G_1U_2\right) \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix} \\ E_2\beta_2(t)\begin{pmatrix}\left(C+\frac{\delta_2}{\lambda_2}L_1+(b-a)F_1J_1\right) \\ \|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ +\left(\frac{\delta_2}{\lambda_2}L_2+(b-a)F_1J_2\right) \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}\end{pmatrix}$$

Therefore

$$\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le$$
$$\begin{pmatrix} A_2\beta_1(t)\begin{pmatrix}\Omega_1\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ +\Omega_2\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix} \\ E_2\beta_2(t)\begin{pmatrix}\Omega_3\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ +\Omega_4\|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}\end{pmatrix}$$

Hence

$$\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le$$
$$\begin{pmatrix}A_2\beta_1(t)\Omega_1 & A_2\beta_1(t)\Omega_2 \\ E_2\beta_2(t)\Omega_3 & E_2\beta_2(t)\Omega_4\end{pmatrix}\begin{pmatrix}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix} \quad (19)$$

Rewrite inequality (19) in a vector form

$$z_{m+1}(t,x_0,y_0) \le Q(t)\, z_m(t,x_0,y_0) \quad (20)$$

where

$$z_{m+1}(t,x_0,y_0) = \begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix}$$

$$Q(t) = \begin{pmatrix}A_2\beta_1(t)\Omega_1 & A_2\beta_1(t)\Omega_2 \\ E_2\beta_2(t)\Omega_3 & E_2\beta_2(t)\Omega_4\end{pmatrix}$$

$$z_m(t,x_0,y_0) = \begin{pmatrix}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}$$

$$z_0 \le \begin{pmatrix}P_1\|x_0\| + A_2\frac{T}{2}(B\|y_0\|+\Lambda_1) \\ P_2\|y_0\| + E_2\frac{T}{2}(C\|x_0\|+\Lambda_2)\end{pmatrix}$$

It follows from inequality (20) that

$$z_{m+1}(t) \le Q_0 z_m(t) \quad (21)$$

where $Q_0 = \max_{t\in[0,T]} \mathrm{Q(t)}$

By iterating inequality (21) we have

$$z_{m+1}(t) \le Q_0^m z_0 \quad (22)$$

which leads to the estimate

$$\sum_{i=1}^m z_i \le \sum_{i=1}^m Q_0^{i-1} z_0 \quad (23)$$

Since the matrix Q_0 has eigenvalue $\lambda_{max}(Q_0) = \frac{1}{2}\left(A_2\frac{T}{2}\Omega_1 + E_2\frac{T}{2}\Omega_4 + \sqrt{\left(A_2\frac{T}{2}\Omega_1 + E_2\frac{T}{2}\Omega_4\right)^2 - 4\left(A_2E_2\frac{T^2}{4}\Omega_1\Omega_4 - A_2E_2\frac{T^2}{4}\Omega_2\Omega_3\right)}\right) < 1$, the series (23) is uniformly convergent, i.e.

$$\lim_{m\to\infty} \sum_{i=1}^m Q_0^{i-1} z_0 = \sum_{i=1}^\infty Q_0^{i-1} z_0 = (I-Q_0)^{-1} z_0 \quad (24)$$

Let

$$\left.\begin{aligned}\lim_{m\to\infty} x_m(t,x_0,y_0) = x(t,x_0,y_0) \\ \lim_{m\to\infty} y_m(t,x_0,y_0) = y(t,x_0,y_0)\end{aligned}\right\} \quad (25)$$

By inequality (20), the estimate

$$\begin{pmatrix}\|x(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le Q_0^m (I-Q_0)^{-1} z_0 \quad (26)$$

is hold for all $m = 0,1,2,\dots$. ■

III. EXISTENCE OF PERIODIC SOLUTION OF (VF1) AND (VF2).

The problem of the existence of a periodic solution of a period T of equations (VF1) and (VF2) is uniquely connected with that of the existence of zero of the functions $\Delta(0,x_0)$ and $\Delta(0,y_0)$ which have the forms:-

$$\left.\begin{aligned} &\Delta(0,x_0) = Ax_0 + \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)} \begin{pmatrix} By(s,x_0,y_0) + \int_{-\infty}^T K(T,s) \\ f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds \\ +\int_a^b G(T,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\end{pmatrix} ds \\ &\Delta(0,y_0) = Ey_0 + \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)} \begin{pmatrix} Cx(s,x_0,y_0) + \\ \int_{-\infty}^T \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds \\ +\int_a^b \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\end{pmatrix} ds\end{aligned}\right\}, \quad (27)$$

$$\left.\begin{aligned} &\Delta_{\mathrm{m}}(0,x_0) = Ax_0 + \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)} \begin{pmatrix} By_m(s,x_0,y_0) + \int_{-\infty}^T K(T,s) \\ f\left(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\right)ds \\ +\int_a^b G(T,s)g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds\end{pmatrix} ds \\ &\Delta_{\mathrm{m}}(0,y_0) = Ey_0 + \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)} \begin{pmatrix} Cx_m(s,x_0,y_0) + \\ \int_{-\infty}^T \varphi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds + \\ \int_a^b \psi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0))ds\end{pmatrix} ds\end{aligned}\right\} \quad (28)$$

where $m = 0,1,2,\dots$ ■

Theorem4. Under the hypothesis and all conditions of theorem3, the following inequality:

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le \langle \begin{pmatrix} Q_3A_2\Omega_1 \\ Q_3A_2\Omega_2 \end{pmatrix}, Q_0^m(I-Q_0)^{-1}z_0\rangle = d_{1m} \quad (29)$$

$$\|\Delta(0,y_0)-\Delta_m(0,y_0)\| \le \langle \begin{pmatrix} Q_4E_2\Omega_3 \\ Q_4E_2\Omega_4 \end{pmatrix}, Q_0^m(I-Q_0)^{-1}z_0\rangle = d_{2m} \quad (30)$$

i. e.

$$\begin{pmatrix} \|\Delta(0,x_0)-\Delta_m(0,x_0)\| \\ \|\Delta(0,y_0)-\Delta_m(0,y_0)\| \end{pmatrix} \le \begin{pmatrix} Q_3A_2\Omega_1 & Q_3A_2\Omega_2 \\ Q_4E_2\Omega_3 & Q_4E_2\Omega_4 \end{pmatrix} Q_0^m(I-Q_0)^{-1}z_0 ,$$

is hold for all $m \ge 0$,

where $Q_3 = \frac{\|A\|T}{e^{\|A\|T}-\|I\|}$ and $Q_4 = \frac{\|E\|T}{e^{\|E\|T}-\|I\|}$

Proof. By using the relations (27) and (28), we get

$$\begin{pmatrix} \|\Delta(0,x_0)-\Delta_m(0,x_0)\| \\ \|\Delta(0,y_0)-\Delta_m(0,y_0)\| \end{pmatrix} \le \left(\begin{array}{c} \frac{\|A\|}{e^{\|A\|T}-\|I\|}\int_0^T \left\|e^{A(T-s)}\right\| \begin{pmatrix} B\|y(s,x_0,y_0)-y_m(s,x_0,y_0)\| + \int_{-\infty}^T \|K(T,s)\| \\ \left\| \begin{matrix} f(s,x(s,x_0,y_0),y(s,x_0,y_0)) \\ -f(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) \end{matrix} \right\| ds + \\ \int_a^b \|G(T,s)\| \\ \left\| \begin{matrix} g(s,x(s,x_0,y_0),y(s,x_0,y_0)) \\ -g(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) \end{matrix} \right\| ds \end{pmatrix} ds \\ \frac{\|E\|}{e^{\|E\|T}-\|I\|}\int_0^T \left\|e^{E(T-s)}\right\| \begin{pmatrix} C\|x(s,x_0,y_0)-x_m(s,x_0,y_0)\| + \\ \int_{-\infty}^T \left\| \begin{matrix} \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0)) \\ -\varphi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) \end{matrix} \right\| ds \\ + \int_a^b \left\| \begin{matrix} \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0)) \\ -\psi(T,s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)) \end{matrix} \right\| ds \end{pmatrix} ds \end{array} \right)$$

Thus

$$\begin{pmatrix} \|\Delta(0,x_0)-\Delta_m(0,x_0)\| \\ \|\Delta(0,y_0)-\Delta_m(0,y_0)\| \end{pmatrix} \le \begin{pmatrix} Q_3A_2 \begin{pmatrix} \left(\frac{\delta_1}{\lambda_1}V_1 + (b-a)G_1U_1\right) \\ \|x(t,x_0,y_0)-x_m(t,x_0,y_0)\| + \\ \left(B+\frac{\delta_1}{\lambda_1}V_2 + (b-a)G_1U_2\right) \\ \|y(t,x_0,y_0)-y_m(t,x_0,y_0)\| \end{pmatrix} \\ Q_4E_2 \begin{pmatrix} \left(C+\frac{\delta_2}{\lambda_2}L_1 + (b-a)F_1J_1\right) \\ \|x(t,x_0,y_0)-x_m(t,x_0,y_0)\| + \\ \left(\frac{\delta_2}{\lambda_2}L_2 + (b-a)F_1J_2\right) \\ \|y(t,x_0,y_0)-y_m(t,x_0,y_0)\| \end{pmatrix} \end{pmatrix}$$

So that

$$\begin{pmatrix} \|\Delta(0,x_0)-\Delta_m(0,x_0)\| \\ \|\Delta(0,y_0)-\Delta_m(0,y_0)\| \end{pmatrix} \le \begin{pmatrix} Q_3A_2 \begin{pmatrix} \Omega_1\|x(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ +\Omega_2\|y(t,x_0,y_0)-y_m(t,x_0,y_0)\| \end{pmatrix} \\ Q_4E_2 \begin{pmatrix} \Omega_3\|x(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ +\Omega_4\|y(t,x_0,y_0)-y_m(t,x_0,y_0)\| \end{pmatrix} \end{pmatrix}$$

Therefore

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\| \le \langle \begin{pmatrix} Q_3A_2\Omega_1 \\ Q_3A_2\Omega_2 \end{pmatrix}, Q_0^m(I-Q_0)^{-1}z_0\rangle = d_{1m}$$

$$\|\Delta(0,y_0)-\Delta_m(0,y_0)\| \le \langle \begin{pmatrix} Q_4E_2\Omega_3 \\ Q_4E_2\Omega_4 \end{pmatrix}, Q_0^m(I-Q_0)^{-1}z_0\rangle = d_{2m} . \blacksquare$$

By using theorem4, we can state and prove the following theorem.

Theorem5. Let the system of equations (VF1) and (VF2) be defined on an interval $a \le x \le b$ and $c \le y \le d$ of a straight line R^1. Assume that for a real t and an integral $m \ge 1$ the function (28) satisfies the inequalities

$$\left.\begin{array}{l} \min\limits_{a+h_1\le x\le b-h_1} \Delta_m(0,x) \le -d_{1m} \\ \max\limits_{a+h_1\le x\le b-h_1} \Delta_m(0,x) \ge d_{1m} \end{array}\right\} \quad (31)$$

$$\left.\begin{array}{l} \min\limits_{c+h_2\le y\le d-h_2} \Delta_m(0,y) \le -d_{2m} \\ \max\limits_{c+h_2\le y\le d-h_2} \Delta_m(0,y) \ge d_{2m} \end{array}\right\} \quad (32)$$

where $h_1 = P_1\|x_0\| + \beta_1(t)A_2[B\|y_0\| + \Lambda_1]$ and $h_2 = P_2\|y_0\| + \beta_2(t)E_2[C\|x_0\| + \Lambda_2]$

Then the equations (VF1) and (VF2) have a periodic solution $x = x(t,x_0,y_0)$ and $y = y(t,x_0,y_0)$ for which $a+h_1 \le x \le b-h_1$ and $c+h_2 \le y \le d-h_2$.

Proof. Let x_1, x_2 and y_1, y_2 are points in the interval $[a+h_1 \le x \le b-h_1]$ and $[c+h_2 \le y \le d-h_2]$ such that:

$$\left.\begin{array}{l} \Delta_m(0,x_1) = \min\limits_{a+h_1\le x\le b-h_1} \Delta_m(0,x) \\ \Delta_m(0,x_2) = \max\limits_{a+h_1\le x\le b-h_1} \Delta_m(0,x) \end{array}\right\} \quad (33)$$

and

$$\left.\begin{array}{l} \Delta_m(0,y_1) = \min\limits_{c+h_2\le y\le d-h_2} \Delta_m(0,y) \\ \Delta_m(0,y_2) = \max\limits_{c+h_2\le y\le d-h_2} \Delta_m(0,y) \end{array}\right\} \quad (34)$$

From the inequalities (31) and (33), we have

$$\left.\begin{array}{l} \Delta(0,x_1) = \Delta_m(0,x_1) + \left(\Delta(0,x_1)-\Delta_m(0,x_1)\right) \le 0 \\ \Delta(0,x_2) = \Delta_m(0,x_2) + \left(\Delta(0,x_2)-\Delta_m(0,x_2)\right) \ge 0 \end{array}\right\} \quad (35)$$

and from the inequalities (32) and (34), we have

$$\left.\begin{array}{l}\Delta(0,y_1)=\Delta_m(0,y_1)+\big(\Delta(0,y_1)-\Delta_m(0,y_1)\big)\le 0\\ \Delta(0,y_2)=\Delta_m(0,y_2)+\big(\Delta(0,y_2)-\Delta_m(0,y_2)\big)\ge 0\end{array}\right\}(36)$$

It follows from (35) and (36) in virtue of the continuity of the Δ-constant that there exist a point x and y, then $x\in[x_1,x_2], y\in[y_1,y_2]$, such that $\Delta(0,x)=0$ and $\Delta(0,y)=0$ .This means that $x=x(t,x_0,y_0)$ and $y=y(t,x_0,y_0)$ are periodic solutions for

$x\in(a+h_1,b-h_1)$ and $y\in(c+h_2,d-h_2)$. ■

Theorem6. Let the vector functions $f(t,x,y)$, $g(t,x,y)$, $\varphi(t,s,x,y)$ and $\psi(t,s,x,y)$ are defined, continuous and periodic in t of period T on the domain (1) and satisfying the following conditions of theorem3 and all the above functions are odd .

Then the solutions $x=x(t,x_0,y_0)$ and $y=y(t,x_0,y_0)$ of (VF1) and (VF2) for which $x(0)\in D_f$ and $y(0)\in D_{1f}$ are periodic in t of period T.

Proof. Given $x_0\in D_f$ and $y_0\in D_{1f}$, consider the successive approximations

$$\begin{pmatrix}x_{m+1}(t,x_0,y_0)\\ y_{m+1}(t,x_0,y_0)\end{pmatrix}=$$
$$\begin{pmatrix}(e^{At}-At)x_0+e^{A(t-s)}\\ \int_0^t\begin{pmatrix}\begin{pmatrix}By_m(s,x_0,y_0)+\int_{-\infty}^t K(t,s)\\ f\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)ds\\ +\int_a^b G(t,s)\\ g\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)ds\end{pmatrix}\\ -\frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\\ \begin{pmatrix}By_m(s,x_0,y_0)+\\ \int_{-\infty}^T K(T,s)\\ f\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)ds\\ +\int_a^b G(T,s)\\ g\big(s,x_m(s,x_0,y_0),y_m(s,x_0,y_0)\big)ds\end{pmatrix}ds\end{pmatrix}ds\\ (e^{Et}-Et)y_0+e^{E(t-s)}\\ \int_0^t\begin{pmatrix}\begin{pmatrix}Cx_m(s,x_0,y_0)+\\ \int_{-\infty}^t\varphi\begin{pmatrix}t,s,x_m(s,x_0,y_0)\\ ,y_m(s,x_0,y_0)\end{pmatrix}ds+\\ \int_a^b\psi\begin{pmatrix}t,s,x_m(s,x_0,y_0)\\ ,y_m(s,x_0,y_0)\end{pmatrix}ds\end{pmatrix}\\ -\frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\\ \begin{pmatrix}Cx_m(s,x_0)+\\ \int_{-\infty}^T\varphi\begin{pmatrix}T,s,x_m(s,x_0,y_0),\\ y_m(s,x_0,y_0)\end{pmatrix}ds\\ +\int_a^b\psi\begin{pmatrix}T,s,x_m(s,x_0,y_0)\\ ,y_m(s,x_0,y_0)\end{pmatrix}ds\end{pmatrix}ds\end{pmatrix}ds\end{pmatrix}\quad(37)$$

$m=0,1,2,\ldots\ldots$

Since $f(t,x_0,y_0)$ is an odd function, $\Delta(0,x)=0$ and $\Delta(0,y)=0$.

Then

$$\begin{pmatrix}x_1(t,x_0,y_0)\\ y_1(t,x_0,y_0)\end{pmatrix}=$$
$$\begin{pmatrix}(e^{At}-At)x_0+e^{A(t-s)}\\ \int_0^t\begin{pmatrix}\begin{pmatrix}By_0+\int_{-\infty}^t K(t,s)f(s,x_0,y_0)ds\\ +\int_a^b G(t,s)g(s,x_0,y_0)ds\end{pmatrix}\\ -\frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\\ \begin{pmatrix}By_0++\int_{-\infty}^T K(T,s)f(s,x_0,y_0)ds\\ +\int_a^b G(T,s)g(s,x_0,y_0)ds\end{pmatrix}ds\end{pmatrix}ds\\ (e^{Et}-Et)y_0+e^{E(t-s)}\\ \int_0^t\begin{pmatrix}\begin{pmatrix}Cx_0+\int_{-\infty}^t\varphi(t,s,x_0,y_0)ds\\ +\int_a^b\psi(t,s,x_0,y_0)ds\end{pmatrix}\\ -\frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\\ \begin{pmatrix}Cx_0+\int_{-\infty}^T\varphi(T,s,x_0,y_0)ds\\ +\int_a^b\psi(T,s,x_0,y_0)ds\end{pmatrix}ds\end{pmatrix}ds\end{pmatrix}$$

$$=\begin{pmatrix}x_1(t+T,x_0,y_0)\\ y_1(t+T,x_0,y_0)\end{pmatrix}$$

Hence

$$\begin{pmatrix}x_1(t,x_0,y_0)\\ y_1(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x_0+\int_0^t\Big(f(s,x_0,y_0)-\frac{1}{T}\int_0^T f(s,x_0,y_0)\Big)ds\\ y_0+\int_0^t\Big(f(s,x_0,y_0)-\frac{1}{T}\int_0^T f(s,x_0,y_0)\Big)ds\end{pmatrix}=\begin{pmatrix}x_1(t+T,x_0,y_0)\\ y_1(t+T,x_0,y_0)\end{pmatrix}$$

i. e.

The functions $x_1(t,x_0,y_0)$ and $y_1(t,x_0,y_0)$ are periodic in t of period T .

Moreover,

$$\begin{pmatrix}\|x_1\ (t,x_0,y_0)-x_o\|\\ \|y_1\ (t,x_0,y_0)-y_o\|\end{pmatrix}\le\begin{pmatrix}P_1\|x_0\|+\beta_1(t)A_2[B\|y_0\|+\Lambda_1]\\ P_2\|y_0\|+\beta_2(t)E_2[C\|x_0\|+\Lambda_2]\end{pmatrix}$$

i. e.

The functions $x_1(t,x_0,y_0)\in D$ and $y_1(t,x_0,y_0)\in D_1$.

Thus,

$\begin{pmatrix}x_1(t,x_0,y_0)\\ y_1(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x_1(-t,x_0,y_0)\\ y_1(-t,x_0,y_0)\end{pmatrix}$ as the integral of an odd function

By induction, we get

$$\begin{pmatrix}x_m(t,x_0,y_0)\\ y_m(t,x_0,y_0)\end{pmatrix}=\begin{pmatrix}x_m(-t,x_0,y_0)\\ y_m(-t,x_0,y_0)\end{pmatrix}$$

and the inequality

$$\begin{pmatrix}\|x_m(t,x_0,y_0)-x_o\| \\ \|y_m(t,x_0,y_0)-y_o\|\end{pmatrix} \le \begin{pmatrix}P_1\|x_0\|+\beta_1(t)A_2[B\|y_0\|+\Lambda_1] \\ P_2\|y_0\|+\beta_2(t)E_2[C\|x_0\|+\Lambda_2]\end{pmatrix}$$

for all $m \ge 1$. ■

Theorem7. Let the system of equations (VF1) and (VF2) be given in the domain D and D_1 Suppose that D_2 and D_3 are a set belonging D_f and D_{1f} respectively then, for D_2 and D_3 to have a point at which $\Delta(0,x)=0$ and $\Delta(0,y)=0$ are zero. Then

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} \le \begin{pmatrix} A\|x_0\|+Q_3A_2 \\ \left[B\|y(s,x_0,y_0)\|+\frac{\delta_1}{\lambda_1}M_1+(b-a)G_1M_2\right] \\ E\|y_0\|+Q_4E_2 \\ \left[C\|x(s,x_0,y_0)\|+\frac{\delta_2}{\lambda_2}N_1+(b-a)N_2\right]\end{pmatrix} + \begin{pmatrix}Q_3A_2\Omega_1 & Q_3A_2\Omega_2 \\ Q_4E_2\Omega_3 & Q_4E_2\Omega_4\end{pmatrix} Q_0^m(I-Q_0)^{-1}z_0 .$$

for all $m \ge 0$ and $x_1 \in D_2$, $y_1 \in D_3$.

Proof. Let $\Delta(0,x)$ and $\Delta(0,y)$ at the points $x_1 \in D_2, y_1 \in D_3$ be zero, since

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} = \begin{pmatrix}\|\Delta_m(0,x_1)-\Delta(0,x_1)+\Delta(0,x_1)\| \\ \|\Delta_m(0,y_1)-\Delta(0,y_1)+\Delta(0,y_1)\|\end{pmatrix}$$

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} \le \begin{pmatrix}\|\Delta(0,x_1)\| \\ \|\Delta(0,y_1)\|\end{pmatrix} + \begin{pmatrix}\|\Delta_m(0,x_1)-\Delta(0,x_1)\| \\ \|\Delta_m(0,y_1)-\Delta(0,y_1)\|\end{pmatrix}$$

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} \le \begin{pmatrix}\|\Delta(0,x_1)\| \\ \|\Delta(0,y_1)\|\end{pmatrix} + \begin{pmatrix}Q_3A_2\Omega_1 & Q_3A_2\Omega_2 \\ Q_4E_2\Omega_3 & Q_4E_2\Omega_4\end{pmatrix} Q_0^m(I-Q_0)^{-1}z_0$$

Thus

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} \le \begin{pmatrix} \left\| \begin{matrix} Ax_0+\frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)} \\ \begin{pmatrix} By(s,x_0,y_0)+ \\ \int_{-\infty}^T K(t,s)f\big(s,x(s,x_0),y(s,x_0)\big)ds \\ +\int_a^b G(t,s)g\big(s,x(s,x_0),y(s,x_0)\big)ds \end{pmatrix} ds \end{matrix} \right\| \\ \left\| \begin{matrix} Ey_0+\frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)} \\ \begin{pmatrix} Cx_m(s,x_0,y_0)+ \\ \int_{-\infty}^T \varphi\big(t,s,x(s,x_0),y(s,x_0)\big)ds \\ +\int_a^b \psi\big(t,s,x(s,x_0),y(s,x_0)\big)ds \end{pmatrix} ds \end{matrix} \right\| \end{pmatrix} + \begin{pmatrix}Q_3A_2\Omega_1 & Q_3A_2\Omega_2 \\ Q_4E_2\Omega_3 & Q_4E_2\Omega_4\end{pmatrix} Q_0^m(I-Q_0)^{-1}Z_0$$

Hence

$$\begin{pmatrix}\|\Delta_m(0,x_1)\| \\ \|\Delta_m(0,y_1)\|\end{pmatrix} \le \begin{pmatrix} A\|x_0\|+Q_3A_2\left[B\|y(s,x_0,y_0)\|+\frac{\delta_1}{\lambda_1}M_1+(b-a)G_1M_2\right] \\ E\|y_0\|+Q_4E_2\left[C\|y(s,x_0,y_0)\|+\frac{\delta_2}{\lambda_2}N_1+(b-a)N_2\right]\end{pmatrix} + \begin{pmatrix}Q_3A_2\Omega_1 & Q_3A_2\Omega_2 \\ Q_4E_2\Omega_3 & Q_4E_2\Omega_4\end{pmatrix} Q_0^m(I-Q_0)^{-1}z_0 . \qquad (38)$$

III. STABILITY SOLUTION OF (VF1) AND (VF2).

The study of the stability solution of the equations (VF1) and (VF2) will be introduced by the theorem shown below:

Theorem8. If the functions $\Delta(0,x_0)$ and $\Delta(0,y_0)$ are defined by (27), where $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are the limit of a sequence of periodic functions (12) and (13), then the following inequalities are holds:

$$\begin{pmatrix}\|\Delta_1(0,x_0^1,y_0^1)-\Delta_1(0,x_0^2,y_0^2)\| \\ \|\Delta_2(0,x_0^1,y_0^1)-\Delta_2(0,x_0^2,y_0^2)\|\end{pmatrix} \le \begin{pmatrix} \|A\|\|x_0^1-x_0^2\|+Q_3A_2 \\ \begin{pmatrix} \Omega_1\begin{bmatrix}\|x_0^1-x_0^2\|F_2F_1(1-E_2B_2(t)\Omega_4)O_1 \\ +A_2B_1(t)\Omega_2\|y_0^1-y_0^2\|F_2F_1O_2\end{bmatrix}+ \\ \Omega_2\begin{bmatrix}\|y_0^1-y_0^2\|F_3F_1(1-A_2B_1(t)\Omega_1)O_2 \\ +E_2B_2(t)\Omega_3\|x_0^1-x_0^2\|F_3F_1O_1\end{bmatrix}\end{pmatrix} \\ \|E\|\|y_0^1-y_0^2\|+Q_4E_2 \\ \begin{pmatrix} \Omega_3\begin{bmatrix}\|x_0^1-x_0^2\|F_2F_1(1-E_2B_2(t)\Omega_4)O_1 \\ +A_2B_1(t)\Omega_2\|y_0^1-y_0^2\|F_2F_1O_2\end{bmatrix}+ \\ \Omega_4\begin{bmatrix}\|y_0^1-y_0^2\|F_3F_1(1-A_2B_1(t)\Omega_1)O_2 \\ +E_2B_2(t)\Omega_3\|x_0^1-x_0^2\|F_3F_1O_1\end{bmatrix}\end{pmatrix}\end{pmatrix}$$

where $O_1=\|e^{At}-At\|$ and $O_2=\|e^{Et}-Et\|$

for all $x_0,x_0^1,x_0^2 \in D$ and $y_0,y_0^1,y_0^2 \in D_1$

Proof. From the equation (27), and by using (28) we can fined

$$\begin{pmatrix}\|\Delta_1(0,x_0^1,y_0^1)-\Delta_1(0,x_0^2,y_0^2)\| \\ \|\Delta_2(0,x_0^1,y_0^1)-\Delta_2(0,x_0^2,y_0^2)\|\end{pmatrix} \le \begin{pmatrix} \|A\|\|x_0^1-x_0^2\|+\frac{\|A\|}{e^{\|A\|T}-\|I\|}\int_0^T\left\|e^{A(T-s)}\right\| \\ \begin{pmatrix} \|B\|\|y(s,x_0^1,y_0^1)-y(s,x_0^2,y_0^2)\|+\frac{\delta_1}{\lambda_1} \\ \begin{pmatrix}V_1\|x(s,x_0^1,y_0^1)-x(s,x_0^2,y_0^2)\|+ \\ V_2\|y(s,x_0^1,y_0^1)-y(s,x_0^2,y_0^2)\|\end{pmatrix}+ \\ (b-a)G_1\begin{pmatrix}U_1\|x(s,x_0^1,y_0^1)-x(s,x_0^2,y_0^2)\| \\ +U_2\|y(s,x_0^1,y_0^1)-y(s,x_0^2,y_0^2)\|\end{pmatrix}\end{pmatrix} ds \\ \|E\|\|y_0^1-y_0^2\|+\frac{\|E\|}{e^{\|E\|T}-\|I\|}\int_0^T\left\|e^{E(T-s)}\right\| \\ \begin{pmatrix} \|C\|\|x(s,x_0^1,y_0^1)-x(s,x_0^2,y_0^2)\|+\frac{\delta_2}{\lambda_2} \\ \begin{pmatrix}L_1\|x(s,x_0^1,y_0^1)-x(s,x_0^2,y_0^2)\| \\ +L_2\|y(s,x_0^1,y_0^1)-y(s,x_0^2,y_0^2)\|\end{pmatrix}+ \\ (b-a)F_1\begin{pmatrix}J_1\|x(s,x_0^1,y_0^1)-x(s,x_0^2,y_0^2)\| \\ +J_2\|y(s,x_0^1,y_0^1)-y(s,x_0^2,y_0^2)\|\end{pmatrix}\end{pmatrix} ds \end{pmatrix}$$

$$\le \begin{pmatrix} \|A\|\|x_0^1 - x_0^2\| + Q_3 A_2 \\ \begin{pmatrix} \left(\frac{\delta_1}{\lambda_1} V_1 + (b-a) G_1 U_1\right) \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + \\ \left(B + \frac{\delta_1}{\lambda_1} V_2 + (b-a) G_1 U_2\right) \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \\ \|E\|\|y_0^1 - y_0^2\| + Q_4 E_2 \\ \begin{pmatrix} \left(C + \frac{\delta_2}{\lambda_2} L_1 + (b-a) F_1 J_1\right) \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + \\ \left(\frac{\delta_2}{\lambda_2} L_2 + (b-a) F_1 J_2\right) \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \end{pmatrix}$$

$$\le \begin{pmatrix} \|A\|\|x_0^1 - x_0^2\| + Q_3 A_2 \begin{pmatrix} \Omega_1 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ +\Omega_2 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \\ \|E\|\|y_0^1 - y_0^2\| + Q_4 E_2 \begin{pmatrix} \Omega_3 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ +\Omega_4 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \end{pmatrix} \qquad (39)$$

where $x(t,x_o^1,y_0^1), x(t,x_0^2,y_0^2), y(t,x_o^1,y_0^1), y(t,x_0^2,y_0^2)$ are the solutions of the integral equation :

$$x(t,x_0^k,y_0^k) = (e^{At} - At)x_0^k + \int_0^t \Big(e^{A(t-s)} \Big(By(s,x_0^k,y_0^k) + \int_{-\infty}^t K(t,s) f(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k))ds + \int_a^b G(t,s) g(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds\Big) - \frac{A}{e^{AT}-I} \int_0^T e^{A(T-s)} \Big(By(s,x_0^k,y_0^k) + \int_{-\infty}^T K(T,s) f(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds + \int_a^b G(T,s) g(s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds\Big) ds\Big) ds, \quad (40)$$

and

$$y(t,x_0^k,y_0^k) = (e^{Et} - Et)y_0^k + \int_0^t \Big(e^{E(t-s)} \Big(Cx(s,x_0^k,y_0^k) + \int_{-\infty}^t \varphi(t,s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds + \int_a^b \psi(t,s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds\Big) - \frac{E}{e^{ET}-I} \int_0^T e^{E(T-s)} \Big(Cx(s,x_0^k,y_0^k) + \int_{-\infty}^T \varphi(T,s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds + \int_a^b \psi(T,s,x(s,x_0^k,y_0^k),y(s,x_0^k,y_0^k)ds\Big) ds\Big) ds, \qquad (41)$$

where $k = 1,2$.

Now, by using (40) and (41), we have

$$\begin{pmatrix} \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \le$$

$$\begin{pmatrix} \|x_0^1 - x_0^2\|\|e^{At} - At\| + \left[\|I\| - \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|I\|}\right)\right] \int_0^t \|e^{A(t-s)}\| \begin{pmatrix} B\|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| + \frac{\delta_1}{\lambda_1} \\ \begin{pmatrix} V_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +V_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| \end{pmatrix} \\ +(b-a) G_1 (U_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +U_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\|) \end{pmatrix} ds \\ + \left[\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|I\|}\right] \int_t^T \|e^{A(T-s)}\| \begin{pmatrix} B\|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| + \frac{\delta_1}{\lambda_1} \\ \begin{pmatrix} V_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +V_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| \end{pmatrix} \\ +(b-a) G_1 (U_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +U_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\|) \end{pmatrix} ds \\ \|y_0^1 - y_0^2\|\|e^{Et} - Et\| + \left[\|I\| - \left(\frac{e^{\|E\|T} - e^{\|E\|(T-t)}}{e^{\|E\|T} - \|I\|}\right)\right] \int_0^t \|e^{E(t-s)}\| \begin{pmatrix} C\|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| + \frac{\delta_2}{\lambda_2} \\ \begin{pmatrix} L_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| + \\ L_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| \end{pmatrix} \\ +(b-a) F_1 (J_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +J_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\|) \end{pmatrix} ds \\ + \left[\frac{e^{\|E\|T} - e^{\|E\|(T-t)}}{e^{\|E\|T} - \|I\|}\right] \int_t^T \|e^{E(T-s)}\| \begin{pmatrix} C\|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| + \frac{\delta_2}{\lambda_2} \\ \begin{pmatrix} L_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| + \\ L_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\| \end{pmatrix} \\ +(b-a) F_1 (J_1 \|x(s,x_0^1,y_0^1) - x(s,x_0^2,y_0^2)\| \\ +J_2 \|y(s,x_0^1,y_0^1) - y(s,x_0^2,y_0^2)\|) \end{pmatrix} ds \end{pmatrix}$$

$$\le \begin{pmatrix} \|x_0^1 - x_0^2\| O_1 + A_2 \beta_1(t) \\ \begin{pmatrix} \left(\frac{\delta_1}{\lambda_1} V_1 + (b-a) G_1 U_1\right) \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + \\ (B + \frac{\delta_1}{\lambda_1} V_2 + (b-a) G_1 U_2) \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \\ \|y_0^1 - y_0^2\| O_2 + E_2 \beta_2(t) \\ \begin{pmatrix} \left(C + \frac{\delta_2}{\lambda_2} L_1 + (b-a) F_1 J_1\right) \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| + \\ (\frac{\delta_2}{\lambda_2} L_2 + (b-a) F_1 J_2) \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \end{pmatrix}$$

$$\le \begin{pmatrix} \|x_0^1 - x_0^2\| O_1 + A_2 \beta_1(t) \Omega_1 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ +A_2 \beta_1(t) \Omega_2 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \\ \|y_0^1 - y_0^2\| O_2 + E_2 \beta_2(t) \Omega_3 \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ +E_2 \beta_2(t) \Omega_4 \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix}$$

$$\leq \begin{pmatrix} \|x_0^1 - x_0^2\|(1 - A_2\beta_1(t)\Omega_1)^{-1}O_1 + A_2\beta_1(t)\Omega_2 \\ (1 - A_2\beta_1(t)\Omega_1)^{-1}\|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \\ \|y_0^1 - y_0^2\|(1 - E_2\beta_2(t)\Omega_4)^{-1}O_2 + E_2\beta_2(t)\Omega_3 \\ (1 - E_2\beta_2(t)\Omega_4)^{-1}\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \end{pmatrix}$$

By substituting the above inequalities, we get

$$\begin{pmatrix} \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \leq \begin{pmatrix} \|x_0^1 - x_0^2\|(1 - A_2\beta_1(t)\Omega_1)^{-1}O_1 + \\ A_2\beta_1(t)\Omega_2(1 - A_2\beta_1(t)\Omega_1)^{-1} \\ \begin{bmatrix} \|y_0^1 - y_0^2\|(1 - E_2\beta_2(t)\Omega_4)^{-1}O_2 + E_2\beta_2(t)\Omega_3 \\ (1 - E_2\beta_2(t)\Omega_4)^{-1}\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \end{bmatrix} \\ \|y_0^1 - y_0^2\|(1 - E_2\beta_2(t)\Omega_4)^{-1}O_2 + \\ E_2\beta_2(t)\Omega_3(1 - E_2\beta_2(t)\Omega_4)^{-1} \\ \begin{bmatrix} \|x_0^1 - x_0^2\|(1 - A_2\beta_1(t)\Omega_1)^{-1}O_1 + A_2\beta_1(t)\Omega_2 \\ (1 - A_2\beta_1(t)\Omega_1)^{-1}\|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{bmatrix} \end{pmatrix}$$

$$\leq \begin{pmatrix} \|x_0^1 - x_0^2\|(1 - A_2\beta_1(t)\Omega_1)^{-1}O_1 + A_2\beta_1(t)\Omega_2\|y_0^1 - y_0^2\| \\ [(1 - A_2\beta_1(t)\Omega_1)(1 - E_2\beta_2(t)\Omega_4)]^{-1}O_2 \\ +E_2\beta_2(t)\Omega_3A_2\beta_1(t)\Omega_2 \\ [(1 - A_2\beta_1(t)\Omega_1)(1 - E_2\beta_2(t)\Omega_4)]^{-1} \\ \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ \|y_0^1 - y_0^2\|(1 - E_2\beta_2(t)\Omega_4)^{-1}O_2 + E_2\beta_2(t)\Omega_3\|x_0^1 - x_0^2\| \\ [(1 - A_2\beta_1(t)\Omega_1)(1 - E_2\beta_2(t)\Omega_4)]^{-1}O_1 \\ +A_2\beta_1(t)\Omega_2E_2\beta_2(t)\Omega_3 \\ [(1 - A_2\beta_1(t)\Omega_1)(1 - E_2\beta_2(t)\Omega_4)]^{-1} \\ \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix}$$

Putting $F_1 = [(1 - A_2\beta_1(t)\Omega_1)(1 - E_2\beta_2(t)\Omega_4)]^{-1}$, then $F_1(1 - A_2\beta_1(t)\Omega_1) = (1 - E_2\beta_2(t)\Omega_4)^{-1}$ and $F_1(1 - E_2\beta_2(t)\Omega_4) = (1 - A_2\beta_1(t)\Omega_1)^{-1}$

So

$$\begin{pmatrix} \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \leq \begin{pmatrix} \|x_0^1 - x_0^2\|(1 - E_2\beta_2(t)\Omega_3A_2\beta_1(t)\Omega_2F_1)^{-1} \\ F_1(1 - E_2\beta_2(t)\Omega_4)O_1 + A_2\beta_1(t)\Omega_2\|y_0^1 - y_0^2\| \\ (1 - E_2\beta_2(t)\Omega_3A_2\beta_1(t)\Omega_2F_1)^{-1}F_1O_2 \\ \|y_0^1 - y_0^2\|(1 - A_2\beta_1(t)\Omega_2E_2\beta_2(t)\Omega_3F_1)^{-1} \\ F_1(1 - A_2\beta_1(t)\Omega_1)O_2 + E_2\beta_2(t)\Omega_3\|x_0^1 - x_0^2\| \\ (1 - A_2\beta_1(t)\Omega_2E_2\beta_2(t)\Omega_3F_1)^{-1}F_1O_1 \end{pmatrix}$$

Putting $F_2 = (1 - E_2\beta_2(t)\Omega_3A_2\beta_1(t)\Omega_2F_1)^{-1}$ and $F_3 = (1 - A_2\beta_1(t)\Omega_2E_2\beta_2(t)\Omega_3F_1)^{-1}$,then

$$\begin{pmatrix} \|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\| \\ \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\| \end{pmatrix} \leq \begin{pmatrix} \|x_0^1 - x_0^2\|F_2F_1(1 - E_2\beta_2(t)\Omega_4)O_1 \\ +A_2\beta_1(t)\Omega_2\|y_0^1 - y_0^2\|F_2F_1O_2 \\ \|y_0^1 - y_0^2\|F_3F_1(1 - A_2\beta_1(t)\Omega_1)O_2 \\ +E_2\beta_2(t)\Omega_3\|x_0^1 - x_0^2\|F_3F_1O_1 \end{pmatrix} \quad (42)$$

So, by substituting the inequalities (42) in (39), we get

$$\begin{pmatrix} \|\Delta_1(0,x_0^1,y_0^1) - \Delta_1(0,x_0^2,y_0^2)\| \\ \|\Delta_2(0,x_0^1,y_0^1) - \Delta_2(0,x_0^2,y_0^2)\| \end{pmatrix} \leq \begin{pmatrix} \|A\|\|x_0^1 - x_0^2\| + Q_3A_2 \\ \begin{pmatrix} \Omega_1\begin{bmatrix} \|x_0^1 - x_0^2\|F_2F_1(1 - E_2B_2(t)\Omega_4)O_1 \\ +A_2B_1(t)\Omega_2\|y_0^1 - y_0^2\|F_2F_1O_2 \end{bmatrix} + \\ \Omega_2\begin{bmatrix} \|y_0^1 - y_0^2\|F_3F_1(1 - A_2B_1(t)\Omega_1)O_2 \\ +E_2B_2(t)\Omega_3\|x_0^1 - x_0^2\|F_3F_1O_1 \end{bmatrix} \end{pmatrix} \\ \|E\|\|y_0^1 - y_0^2\| + Q_4E_2 \\ \begin{pmatrix} \Omega_3\begin{bmatrix} \|x_0^1 - x_0^2\|F_2F_1(1 - E_2B_2(t)\Omega_4)O_1 \\ +A_2B_1(t)\Omega_2\|y_0^1 - y_0^2\|F_2F_1O_2 \end{bmatrix} + \\ \Omega_4\begin{bmatrix} \|y_0^1 - y_0^2\|F_3F_1(1 - A_2B_1(t)\Omega_1)O_2 \\ +E_2B_2(t)\Omega_3\|x_0^1 - x_0^2\|F_3F_1O_1 \end{bmatrix} \end{pmatrix} \end{pmatrix} . \blacksquare$$

4. ANOTHER METHOD

In this section, we prove that the existence and uniqueness theorem of (VF1) and (VF2) by using Banach fixed point theorem.

Theorem9. Let the vector functions $f(t,x,y)$, $g(t,x,y), \varphi(t,s,x,y)$and $\psi(t,s,x,y)$ are defined, continuous and periodic in t of period T on the domain (1) and satisfy all assumptions and conditions of theorem3. Then (VF1) and (VF2) have a unique periodic continuous solution on the domain (1).

Proof: Let $(C[0,T], \|.\|)$ be a Banach space and T^* be a mapping on $C[0,T]$ as follows:

$$T^*x(t,x_0,y_0) = (e^{At} - At)x_0 + \int_0^t \Big(e^{A(t-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^t K(t,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(t,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^T K(T,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(T,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds$$

and

$$T^*y(t,x_0,y_0) = (e^{Et} - Et)y_0 + \int_0^t \Big(e^{E(t-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^t \varphi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^T \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds$$

Since

$$\int_0^t \Big(e^{A(t-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^t K(t,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds +$$

$\int_a^b G(t,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^{T} K(T,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(T,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds$ is continuous on the domain (1) and also

$\int_0^t \Big(e^{E(t-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^{t} \varphi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^{T} \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds$ is continuous on the domain (1).

Therefore: $T^*: C[0,T] \longrightarrow C[0,T]$

Next

Let $x(t,x_0,y_0), z(t,x_0,y_0), y(t,x_0,y_0)$ and $v(t,x_0,y_0)$ are vector functions on $[0,T]$, then

$$\begin{pmatrix} \|T^*x(t,x_0,y_0) - T^*z(t,x_0,y_0)\| \\ \|T^*y(t,x_0,y_0) - T^*v(t,x_0,y_0)\| \end{pmatrix} \le$$

$$\begin{pmatrix} \Big[\|I\| - \Big(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|I\|}\Big)\Big] \\ \int_0^t \|e^{A(t-s)}\| \begin{pmatrix} B\|y(s,x_0,y_0) - v(s,x_0,y_0)\| + \frac{\delta_1}{\lambda_1} \\ \begin{pmatrix} V_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +V_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\| \end{pmatrix} + \\ (b-a)G_1(U_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +U_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\|) \end{pmatrix} ds \\ + \Big[\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|I\|}\Big] \\ \int_t^T \|e^{A(T-s)}\| \begin{pmatrix} B\|y(s,x_0,y_0) - v(s,x_0,y_0)\| + \frac{\delta_1}{\lambda_1} \\ \begin{pmatrix} V_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +V_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\| \end{pmatrix} + \\ (b-a)G_1(U_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +U_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\|) \end{pmatrix} ds \\ \Big[\|I\| - \Big(\frac{e^{\|E\|T} - e^{\|E\|(T-t)}}{e^{\|E\|T} - \|I\|}\Big)\Big] \\ \int_0^t \|e^{E(t-s)}\| \begin{pmatrix} C\|x(s,x_0,y_0) - z(s,x_0,y_0)\| + \frac{\delta_2}{\lambda_2} \\ \begin{pmatrix} L_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +L_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\| \end{pmatrix} + \\ (b-a)F_1(J_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +J_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\|) \end{pmatrix} ds \\ + \Big[\frac{e^{\|E\|T} - e^{\|E\|(T-t)}}{e^{\|E\|T} - \|I\|}\Big] \\ \int_t^T \|e^{E(T-s)}\| \begin{pmatrix} C\|x(s,x_0,y_0) - z(s,x_0,y_0)\| + \frac{\delta_2}{\lambda_2} \\ \begin{pmatrix} L_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +L_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\| \end{pmatrix} + \\ (b-a)F_1(J_1\|x(s,x_0,y_0) - z(s,x_0,y_0)\| \\ +J_2\|y(s,x_0,y_0) - v(s,x_0,y_0)\|) \end{pmatrix} ds \end{pmatrix}$$

and so

$$\begin{pmatrix} \|T^*x(t,x_0,y_0) - T^*z(t,x_0,y_0)\| \\ \|T^*y(t,x_0,y_0) - T^*v(t,x_0,y_0)\| \end{pmatrix} \le \begin{pmatrix} A_2\beta_1(t)\Omega_1 & A_2\beta_1(t)\Omega_2 \\ E_2\beta_2(t)\Omega_3 & E_2\beta_2(t)\Omega_4 \end{pmatrix} \begin{pmatrix} \|x(t,x_0,y_0) - z(t,x_0,y_0)\| \\ \|y(t,x_0,y_0) - v(t,x_0,y_0)\| \end{pmatrix}$$

By the condition (11), then T^* is a contraction mapping on $C[0,T]$.

Thus by Banach fixed point theorem, then there exists a fixed point $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ in $C[0,T]$ such that

$T^*x(t,x_0,y_0) = x(t,x_0,y_0)$ and $T^*y(t,x_0,y_0) = y(t,x_0,y_0)$

Therefore,

$$x(t,x_0,y_0) = (e^{At} - At)x_0 + \int_0^t \Big(e^{A(t-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^{t} K(t,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(t,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{A}{e^{AT}-I}\int_0^T e^{A(T-s)}\Big(By(s,x_0,y_0) + \int_{-\infty}^{T} K(T,s)f(s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b G(T,s)g(s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds$$

And

$$y(t,x_0,y_0) = (e^{Et} - Et)y_0 + \int_0^t \Big(e^{E(t-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^{t} \varphi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(t,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big) - \frac{E}{e^{ET}-I}\int_0^T e^{E(T-s)}\Big(Cx(s,x_0,y_0) + \int_{-\infty}^{T} \varphi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds + \int_a^b \psi(T,s,x(s,x_0,y_0),y(s,x_0,y_0))ds\Big)ds\Big)ds\ .\blacksquare$$

V. CONCLUSION

This paper provided the existence and approximation of the periodic solutions for non-linear integrodifferential equations of Volterra- Friedholm type. The numerical-analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by (Samoilenko, A. M.).

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