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Framelets with Three Generator by Unitary Extension Principle

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ABSTRACT

In this paper we have given a straight forward method to generate compactly supported tight wavelet frames (framelets) for a scaling function based on unitary extension principle (UEP). Here we decompose the refinement mask into polyphase form and then apply UEP. Example are discussed for framelets generated from cardinal B-spline of different order.

Key words: Spline, Frame, FMRA, Wavelet frame

1. INTRODUCTION

Frames are the generalization of the idea of orthonormal bases, which maintain the characteristics of orthonormal bases as well as allow some more flexibility in applications. It offers a reduced and stable way of illustrating a given function. Frames was introduced in 1952 by Duffin and Schaeffer in their paper[1], used frame like a tool for the study of non harmonic Fourier series. In 1985, Daubechies, Meyer and Grossman [2] observed that series expansion of function in $L^2(R)$ can be done with frames. Benedetto [6] provides the tool Frame multiresolution analysis for the construction of wavelet frames and is same as the general multi resolution analysis but the orthonornal bases condition is substituted by frame condition. For the construction of tight wavelet frames from the multiresolution analysis , Ron and Shen [7] introduced a sufficiency condition called Unitary extension principle(UEP). For construction of compactly supported tight wavelet frame, Chui[9] introduce maximum vanishing moment. This paper also describes sibling frames as an expanded notion of tight wavelet frames. Petukhov [3] generated a tight compactly supported wavelet frame with two generator and [4] symmetric framelet with three generator associated with symmetric compactly supported refinable function.

Wavelet frames offer an important framework for distinguishing many high-pass frequency component from low-pass frequency portion. Besides this aspect, wavelet frame's redundant property is known to be useful in restoring knowledge from corrupted and noisy one. It is therefore preferred to shift wavelets in applications with wavelet frames. This paper is arranged as follows. In section 2, we revise some preliminary background defining spline , frame , wavelet and framelets. In section 3, an explicit construction of compactly supported framelets with three generator govern by UEP applied on polyphase form of refinable function is explained and examples are given in section 4. The paper is wind up with conclusion in section 5.

2. PRELIMINARY

In our paper, we will consider function of one variable in space in $L^2(R)$ with inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

The Fourier Transform of $f(x) \in L^2(R)$ is defined as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx$$

For any function $f \in L^2(R)$, the dyadic dilation operator D is defined as

$$Df(x) = 2^{\frac{1}{2}} f(2x)$$

and translation operator is defined as

$$T_a f(x) = f(x-a) \qquad x \in R$$

2.1 B-splines

The cardinal B-spline B_m of order $m \geq 1$ is defined as

$$B_m = B_{m-1} * B_1 = \int_0^1 B_{m-1}(.-t)dt \qquad m \geq 1$$

With $B_1 = \chi_{[0,1)}$. It is known that $suppB_m = [0, m]$ and $B_m(m \geq 2)$ satisfies the recursion formula

$$(m-1)B_m(x) = xB_{m-1}(x) + (m-x)B_{m-1}(x-1), \; x \in R$$

By convolution property, the Fourier Transform of B_m is

$$\widehat{B_m}(\xi) = \int_{-\infty}^{\infty} B_m(x)\, e^{-i\xi x}dx = (\frac{1-e^{-i\xi}}{i\xi})^m$$

The two scale relation for cardinal B-spline of order m is written as

$$B_m(x) = \sum_{k=0}^{m} 2^{-m+1}\binom{m}{k} B_m(2x-k)$$

and its Fourier transform is,

$$\widehat{B_m}(\omega) = P(z)\widehat{B_m}(^{\omega}/_2) \qquad (1)$$

where $P(z) = \frac{1}{2}\sum_{k=0}^{m} 2^{-m+1}\binom{m}{k} z^k = (\frac{1+z}{2})^m$; $z = e^{-i\omega}/_2$

P(z) is called the refinement function of the scaling function.

2.2 Frames

Definition 2.2.1. A sequence of function $\{f_k\}_{k=1}^{\infty}$ in $L^2(R)$ is a frame for $L^2(R)$ if $\exists$ constant $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B||f||^2$$

The constant A,B are the frame bounds. If $A = B$, then the frame is called tight frame and is called Parseval frame if $A = B = 1$.

Definition 2.2.2. A sequence of function $\{f_k\}_{k=1}^{\infty}$ in $L^2(R)$ is a Bessel frame if $\exists$ constant $B > 0$ such that

$$\sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B||f||^2 \qquad \forall\, f \in L^2(R)$$

and B is the Bessel bound of $\{f_k\}_{k=1}^{\infty}$. Hence, we can say that a frame sequence is same as Bessel sequence.

Definition 2.2. 3. Let $\psi \in L^2(R)$ define a function ψ_{jk} by

$$\psi_{jk} = D^j T_k \psi = 2^{\frac{-j}{2}}\psi(2.-k) \qquad j.k \in \mathbb{Z} \qquad (2)$$

Then ψ is called a wavelet, if equation (2) forms an orthonormal basis of $L^2(R)$.

Definition 2.2. 4. Given $\Psi = \{\psi_l\}_{l=1}^{N} \subset L^2(R)$, the affine system can be define as

$$X(\psi) = \{\psi_{ljk}: 1 \leq l \leq N, \quad l, j, k \in \mathbb{Z}\} \subset L^2(R)$$

where $\psi_{ljk} = D^j T_k \psi_l$

the set of function $X(\psi)$ is called a tight wavelet frame of $L^2(R)$ if the following conditions meet

$f \in L^2(R)$ s.t

$$||f||^2_{L^2(R)} = \sum_{g \in X(\psi)} |\langle f, g\rangle|^2 \qquad \forall\, f \in L^2(R)$$

or, $$f = \sum_{g \in X(\psi)} \langle f, g\rangle\, g \qquad \forall\, f \in L^2(R)$$

3. FRAMELETS GENERATED BY UNITARY EXTENSION PRINCIPLE

3.1 Multiresolution analysis (MRA)

MRA is a mechanism to fabricate orthonormal bases of $L^2(R)$ of the form $\{\psi_{jk} = 2^{\frac{-j}{2}}\psi(2.-k): \; j, k \in \mathbb{Z}\}$. It was introduced by Mallat [5]. In 1984, Benedetto and Li [6] introduce Frame MRA(FMRA) which is same as MRA only the orthonormal basis for V_0 is shifted by frame condition.

Definition 3.1.1: A Frame multiresolution analysis of $L^2(R)$ is a nested of subspaces

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

that satisfies the following conditions

1. $\bigcap_{j \in Z} V_j = \{0\}$,
2. $\overline{\bigcup_{j \in Z} V_j} = L^2$,
3. $f(.) \in V_j$ if and only if $f(2^{-j}(.)) \in V_{j+1}$,
4. $\varphi \in V_0$ s.t $\{\varphi(.-k)\}_{k \in Z}$ is a frame for V_0

The function φ is known as the scaling function of the frame MRA. Since $\varphi \in V_0 \subset V_1$, a sequence $(g_k) \in l^2(R)$ exist such that the scaling function satisfies

$$\varphi(x) = 2\sum g_k \varphi(2x-k)$$

this equation is called refinement or dilation equation or two scale difference equation. The Fourier transform of (1) is

$$\hat{\varphi}(\omega) = G\left(\frac{\omega}{2}\right)\hat{\varphi}(\frac{\omega}{2})$$

equivalently, $$\hat{\varphi}(2\omega) = G(\omega)\hat{\varphi}(\omega) \qquad (3)$$

where $$G(\omega) = \frac{1}{2}\sum g_k e^{-i\omega k}$$

$$\Rightarrow G(z) = \frac{1}{2}\sum g_k z^k \quad , \; z = e^{-i\omega}$$

is called two scale symbol or refinement mask of φ and is 2π periodic function. Now we define functions $\psi_1, \psi_2, \ldots, \psi_N$ in subspaces V_1 in terms of fourier transform as

$$\hat{\psi}_l(2\omega) = H_l(\omega)\varphi(\omega) \qquad (4)$$

where $H_l(\omega) = \frac{1}{2}\sum h_{lk} e^{-i\omega}$ for some sequence $\{h_{lk}\} \in l^2(R)$.

3.2 Unitary Extension Principle(UEP)

Ron and Shen [7] introduced UEP. Let $\varphi \in L^2(R)$ be the scaling function defined as (3) and G the corresponding refinement mask. For each $l = 1,2,\dots,N$, define $\psi_l \in L^2(R)$ as (4). Consider a matrix of size $(N+1)\times 2$ defined as

$$M(z) = \begin{bmatrix} G(z) & G(-z) \\ H_1(z) & H_1(-z) \\ \vdots & \vdots \\ H_N(z) & H_N(-z) \end{bmatrix}$$

If M satisfies $M^*M = I$ a.e $z \in T$ (5)

Then $\{\psi_{ljk}: 1 \le l \le N,\ j,k \in Z\}$ is a tight frame for $L^2(R)$. Condition (5) is the well known UEP. This condition is also equivalent to

$$M^*M = I = H^*H + GG^*$$

or $$H^*H = I - GG^* \qquad (6)$$

where $H(z) = \begin{bmatrix} H_1(z) & H_1(-z) \\ \vdots & \vdots \\ H_N(z) & H_N(-z) \end{bmatrix}$ and $G(z) = [G(z) \quad G(-z)]^T$

Theorem 3.2.1: A compactly supported refinable function $\varphi \in L^2(R)$ with $\hat{\varphi}(0) = 1$ and refinement mask G(z), has a tight frame ψ with compact support iff

$$|G(z)|^2 + |G(-z)|^2 \le 1 \qquad \forall\ |z| \le 1 \qquad (7)$$

Proof: Since $H^*H = I - GG^*$

$$= \begin{bmatrix} 1 - |G(z)|^2 & -G(z)\overline{G(-z)} \\ -G(-z)\overline{G(z)} & 1 - |G(-z)|^2 \end{bmatrix}$$

This is a positive semi-definite Hermitian matrix with eigen vectors

$$\lambda = 1 \quad and \quad \lambda = 1 - |G(z)|^2 - |G(-z)|^2$$

Due to positive semi-definite,

$$\lambda = 1 - |G(z)|^2 - |G(-z)|^2 \ge 0$$

or, $$|G(z)|^2 + |G(-z)|^2 \le 1$$

Theorem 3.2.2: A compactly supported refinable function $\varphi \in L^2(R)$ with $\hat{\varphi}(0) = 1$and refinement mask G(z) satisfying (7) , has a compactly supported tight frame $\psi = \{\psi_1, \psi_2, \psi_3\}$.

In [8], the author shows that compactly supported wavelet frame $\psi = \{\psi_1, \psi_2\}$ with two generator associated with $\varphi \in L^2(R)$.To prove this theorem we use polyphase decomposition technique. This technique is used to decompose a system function into polyphase representation. Let define a matrix $P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}$. Then the decomposition of $G(z)$ and $H_l(z)$ with respect to matrix P is

$$G(z) = \frac{1}{\sqrt{2}}[G_1(z^2) + zG_2(z^2)]$$

$$H_l(z) = \frac{1}{\sqrt{2}}[H_{l1}(z^2) + zH_{l2}(z^2)]$$

Proof: With polyphase form the matrix M is written as

$$M = \frac{1}{\sqrt{2}}\begin{bmatrix} G_1(z^2) + zG_2(z^2) & G_1(z^2) - zG_2(z^2) \\ H_{11}(z^2) + zH_{12}(z^2) & H_{11}(z^2) - zH_{12}(z^2) \\ \vdots & \vdots \\ H_{N1}(z^2) + zH_{N2}(z^2) & H_{N1}(z^2) - zH_{N2}(z^2) \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} G_1(z^2) & G_2(z^2) \\ H_{11}(z^2) & H_{12}(z^2) \\ \vdots & \vdots \\ H_{N1}(z^2) & H_{N2}(z^2) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}$$

or, $$M\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & z^{-1} \\ 1 & -z^{-1} \end{bmatrix} = \begin{bmatrix} G_1(z^2) & G_2(z^2) \\ H_{11}(z^2) & H_{12}(z^2) \\ \vdots & \vdots \\ H_{N1}(z^2) & H_{N2}(z^2) \end{bmatrix} = M_{GH}$$

or, $$M_{GH}^*M_{GH} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix} M^*M \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & z^{-1} \\ 1 & -z^{-1} \end{bmatrix} = I_2$$

Therefore, from previous theorem,

$$|G_1(z^2)|^2 + |G_2(z^2)|^2 \le 1$$

putting $z^2 = \mu$ with $|\mu| = 1$ so that, $|G_1(\mu)|^2 + |G_2(\mu)|^2 \le 1$

by Riesz lemma, we can find a Laurent polynomial $G_3(\mu)$ s.t

$$|G_1(\mu)|^2 + |G_2(\mu)|^2 + |G_3(\mu)|^2 = 1 \qquad (8)$$

Let $\hat{G}$ be a column vector $[G_1(\mu), G_2(\mu), G_3(\mu)]^T$

Define
$$\hat{H} = I - \hat{G}\hat{G}^* = \begin{bmatrix} 1 - |G_1(\mu)|^2 & -G_1(\mu)\overline{G_2(\mu)} & -G_1(\mu)\overline{G_3(\mu)} \\ -G_2(\mu)\overline{G_1(\mu)} & 1 - |G_2(\mu)|^2 & -G_2(\mu)\overline{G_3(\mu)} \\ -G_3(\mu)\overline{G_1(\mu)} & -G_3(\mu)\overline{G_2(\mu)} & 1 - |G_3(\mu)|^2 \end{bmatrix}$$

we easily show that, $\hat{H}^*\hat{H} = \hat{H} = I - \hat{G}\hat{G}^*$

$$\hat{G}\hat{G}^* + \hat{H}^*\hat{H} = I_{3\times 3}$$

Restricting this rule to 2×2 matrix by considering

$$\widetilde{H} = \begin{bmatrix} 1-|G_1(\mu)|^2 & -G_1(\mu)\overline{G_2(\mu)} \\ -G_2(\mu)\overline{G_1(\mu)} & 1-|G_2(\mu)|^2 \\ -G_3(\mu)\overline{G_1(\mu)} & -G_3(\mu)\overline{G_2(\mu)} \end{bmatrix} \qquad (9)$$

be the first 3×2 block matrix in $\widehat{H}$ and $\widetilde{G}$ be 2×1 matrix , so that,

$$\widetilde{G}\widetilde{G}^* + \widetilde{H}^*\widetilde{H} = I_{2\times 2} \qquad (10)$$

Multiplying polyphase matrix $\bar{P}^*$ and $\bar{P}$ on the both side of (10), we get,

$$\bar{P}^*(\widetilde{G}\widetilde{G}^* + \widetilde{H}^*\widetilde{H})\bar{P} = \bar{P}^* I_{2\times 2}\bar{P}$$

$$\left(\bar{P}^*\widetilde{G}\right)\left(\bar{P}^*\widetilde{G}\right)^* + \left(\widetilde{H}\bar{P}\right)^*\left(\widetilde{H}\bar{P}\right) = I$$

Clearly, $\left(\bar{P}^*\widetilde{G}\right) = G$

Thus, $$\left(\widetilde{H}\bar{P}\right)^*\left(\widetilde{H}\bar{P}\right) = I - GG^* \qquad (11)$$

Comparing (6) and (10), we get,

$$H = \widetilde{H}\bar{P} \quad i.e \quad H^*H = I - GG^* \qquad (12)$$

That is H satisfies the matrix form of UEP condition. Denote the first column of matrix $H = \widetilde{H}P$ by $H_{l1}(\omega), H_{l2}(\omega)$ and $H_{l3}(\omega)$. These are the desirable Laurent polynomials associated with the framelets $\psi_{l1}, \psi_{l2}, \psi_{l3}$ with scaling function φ.

4. EXAMPLES

Example 4.1 Take the linear B-spline B_2 as the scaling function. The refinement mask for it is $G(\omega) = (\frac{1+e^{-i\omega}}{2})^2 \Leftrightarrow G(z) = (\frac{1+z}{2})^2$.

or, $G(z) = \frac{1}{4}(1 + z^2 + 2z) = \frac{1}{\sqrt{2}}\left[\frac{\sqrt{2}}{4}(1+z^2) + \frac{\sqrt{2}}{2}z\right]$

$$G(z) = \frac{1}{\sqrt{2}}[G_1(z^2) + zG_2(z^2)]$$

The polyphase decomposition is

$$G_1(z^2) = \frac{\sqrt{2}}{4}(1+z^2) \quad and \quad G_2(z^2) = \frac{\sqrt{2}}{2}$$

Since, $|G_1(z^2)|^2 + |G_2(z^2)|^2 + |G_3(z^2)|^2 = 1$

Therefore, $G_3(z^2) = \frac{\sqrt{2}}{4}(1 - z^2)$

From (9),

$$\widetilde{H} = \begin{bmatrix} \frac{1}{8}(6 + e^{2i\omega} + e^{-2i\omega}) & \frac{-1}{4}(1 + e^{-2i\omega}) \\ \frac{-1}{4}(1 + e^{2i\omega}) & \frac{1}{2} \\ \frac{-1}{8}(1 - e^{4i\omega}) & \frac{-1}{2}(1 - e^{2i\omega}) \end{bmatrix}$$

Therefore from (12),

$$H = \widetilde{H}\bar{P} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{8}(6 + e^{2i\omega} + e^{-2i\omega}) & \frac{-1}{4}(1 + e^{-2i\omega}) \\ \frac{-1}{4}(1 + e^{2i\omega}) & \frac{1}{2} \\ \frac{-1}{8}(1 - e^{4i\omega}) & \frac{-1}{4}(1 - e^{2i\omega}) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ e^{i\omega} & -e^{i\omega} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{16}e^{-2i\omega} - \frac{\sqrt{2}}{8}e^{-i\omega} + \frac{3\sqrt{2}}{8} - \frac{\sqrt{2}}{8}e^{i\omega} - \frac{\sqrt{2}}{16}e^{2i\omega} & \frac{\sqrt{2}}{16}e^{-2i\omega} + \frac{\sqrt{2}}{8}e^{-i\omega} + \frac{3\sqrt{2}}{8} + \frac{\sqrt{2}}{8}e^{i\omega} - \frac{\sqrt{2}}{16}e^{2i\omega} \\ -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}e^{i\omega} - \frac{\sqrt{2}}{8}e^{2i\omega} & -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{4}e^{i\omega} - \frac{\sqrt{2}}{8}e^{2i\omega} \\ \frac{-\sqrt{2}}{16} + \frac{\sqrt{2}}{16}e^{4i\omega} + \frac{\sqrt{2}}{16}e^{3i\omega} - \frac{\sqrt{2}}{8}e^{i\omega} & \frac{-\sqrt{2}}{16} + \frac{\sqrt{2}}{16}e^{4i\omega} - \frac{\sqrt{2}}{16}e^{3i\omega} + \frac{\sqrt{2}}{8}e^{i\omega} \end{bmatrix}$$

It is clear that the second column is the shift of first column by π. The first column are the refenement mask or Laurent polynomial $\{H_1, H_2, H_3\}$ for the tight framelet generated from linear B-spline function B_2 ,that is,

$$H_1 = \frac{\sqrt{2}}{16}e^{-2i\omega} - \frac{\sqrt{2}}{8}e^{-i\omega} + \frac{3\sqrt{2}}{8} - \frac{\sqrt{2}}{8}e^{i\omega} - \frac{\sqrt{2}}{16}e^{2i\omega}$$

$$H_2 = -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4}e^{i\omega} - \frac{\sqrt{2}}{8}e^{2i\omega}$$

$$H_3 = \frac{-\sqrt{2}}{16}e^{-2i\omega} + \frac{\sqrt{2}}{4}e^{-i\omega} - \frac{\sqrt{2}}{4}e^{i\omega} + \frac{\sqrt{2}}{16}e^{2i\omega}$$

Therefore, the corresponding B-spline tight framelets $\{\psi_2^1, \psi_2^2, \psi_2^3\}$ are

$$\psi_2^1(x) = \frac{\sqrt{2}}{8}B_2(2x+2) - \frac{\sqrt{2}}{4}B_2(2x+1) + \frac{3\sqrt{2}}{4}B_2(2x) - \frac{\sqrt{2}}{4}B_2(2x-1) - \frac{\sqrt{2}}{8}B_2(2x-2)$$

$$\psi_2^2(x) = -\frac{\sqrt{2}}{4}B_2(2x) + \frac{\sqrt{2}}{2}B_2(2x-1) - \frac{\sqrt{2}}{4}B_2(2x-2)$$

$\psi_2^3(x) = \frac{-\sqrt{2}}{8}B_2(2x+2) + \frac{\sqrt{2}}{2}B_2(2x+1) - \frac{\sqrt{2}}{3}B_2(2x - 1) + \frac{\sqrt{2}}{8}B_2(2x-2)$

Linear (second order) B-spline B_2 and the generated compactly supported tight framelets $\{\psi_2^1, \psi_2^2, \psi_2^3\}$ are demonstrated in Figure.1

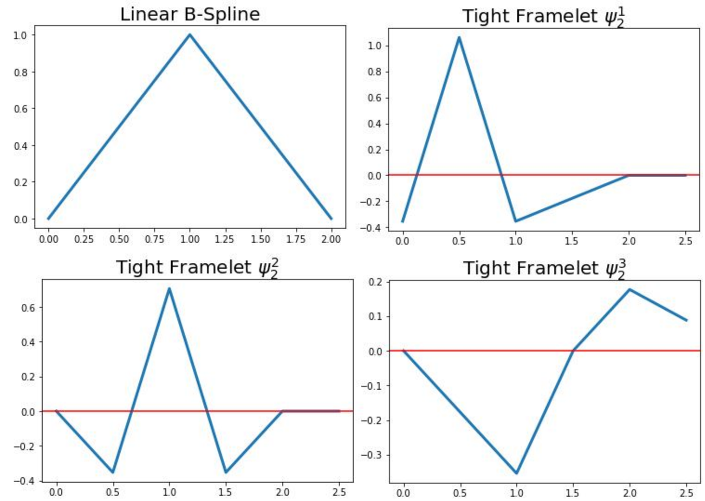

Figure 1: Linear(second order) B-spline and generated tight framelets

Example 4.2 Take the quadratic B-spline B_3 as the scaling function. The refinement mask for it is $G(\omega) = (\frac{1+e^{-i\omega}}{2})^3 \Leftrightarrow G(z) = (\frac{1+z}{2})^3$.

The polyphase decomposition is

$$G_1(z^2) = \frac{\sqrt{2}}{8}(1 + 3z^2) \quad and \quad G_2(z^2) = \frac{\sqrt{2}}{8}(3 + z^2)$$

and $$G_3(z^2) = \frac{3\sqrt{2}}{4}(1 - z^{-2})$$

From(9),

$$\widetilde{H} = \begin{bmatrix} \frac{1}{32}(20 - 3e^{2i\omega} - 3e^{-2i\omega}) & -\frac{1}{32}(6 + e^{2i\omega} + 9e^{-2i\omega}) \\ -\frac{1}{32}(6 + 9e^{2i\omega} + e^{-2i\omega}) & \frac{1}{32}(20 - 3e^{2i\omega} - 3e^{-2i\omega}) \\ \frac{3}{16}(2 + e^{2i\omega} - 3e^{-2i\omega}) & -\frac{3}{16}(2 - 3e^{2i\omega} + e^{-2i\omega}) \end{bmatrix}$$

On solving we get the Laurent polynomial $\{H_1, H_2, H_3\}$ for the tight framelet generated from quadratic B-spline function B_3 as,

$$H_1 = \frac{-3\sqrt{2}}{64}e^{-2i\omega} - \frac{9\sqrt{2}}{64}e^{-i\omega} + \frac{5\sqrt{2}}{16} - \frac{3\sqrt{2}}{32}e^{i\omega} - \frac{3\sqrt{2}}{64}e^{2i\omega} - \frac{\sqrt{2}}{64}e^{3i\omega}$$

$$H_2 = \frac{-\sqrt{2}}{64}e^{-2i\omega} - \frac{3\sqrt{2}}{64}e^{-i\omega} - \frac{3\sqrt{2}}{32} + \frac{5\sqrt{2}}{16}e^{i\omega} - \frac{9\sqrt{2}}{64}e^{2i\omega} - \frac{3\sqrt{2}}{64}e^{3i\omega}$$

$$H_3 = -\frac{3\sqrt{2}}{32} - \frac{9\sqrt{2}}{32}e^{i\omega} - \frac{3\sqrt{2}}{16}e^{2i\omega} + \frac{3\sqrt{2}}{16}e^{3i\omega} + \frac{9\sqrt{2}}{32}e^{4i\omega} + \frac{3\sqrt{2}}{32}e^{5i\omega}$$

Therefore, the corresponding B-spline tight framelets $\{\psi_3^1, \psi_3^2, \psi_3^3\}$ are

$$\psi_3^1(x) = \frac{-3\sqrt{2}}{32}B_3(2x+2) - \frac{9\sqrt{2}}{32}B_3(2x+1) + \frac{5\sqrt{2}}{8}B_3(2x) - \frac{3\sqrt{2}}{16}B_3(2x-1) - \frac{3\sqrt{2}}{32}B_3(2x-2) - \frac{\sqrt{2}}{32}B_3(2x-3)$$

$$\psi_3^2(x) = \frac{-\sqrt{2}}{32}B_3(2x+2) - \frac{3\sqrt{2}}{32}B_3(2x+1) - \frac{3\sqrt{2}}{16}B_3(2x) + \frac{5\sqrt{2}}{8}B_3(2x-1) - \frac{9\sqrt{2}}{32}B_3(2x-2) - \frac{3\sqrt{2}}{32}B_3(2x-3)$$

$$\psi_3^3(x) = -\frac{3\sqrt{2}}{16}B_3(2x) - \frac{9\sqrt{2}}{16}B_3(2x-1) - \frac{3\sqrt{2}}{8}B_3(2x-2) + \frac{3\sqrt{2}}{8}B_3(2x-3) + \frac{9\sqrt{2}}{16}B_3(2x-4) + \frac{3\sqrt{2}}{16}B_3(2x-5)$$

Quadratic (third order) B-spline B_3 and the generated compactly supported tight framelets $\{\psi_3^1, \psi_3^2, \psi_3^3\}$ are demonstrated in Figure.2

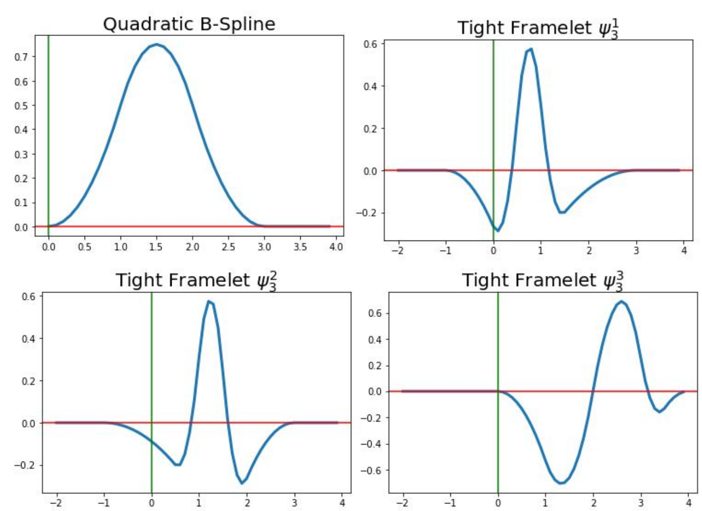

Figure 2: Quadratic(third order) B-spline and generated tight framelets

Example 4.3 Take the cubic B-spline B_4 as the scaling function. The refinement mask for it is $G(\omega) = (\frac{1+e^{-i\omega}}{2})^4 \Leftrightarrow G(z) = (\frac{1+z}{2})^4$.

The polyphase decomposition is

$$G_1(z^2) = \frac{\sqrt{2}}{16}(1 + 6z^2 + z^4) \quad and \quad G_2(z^2) = \frac{\sqrt{2}}{4}(1 + z^2)$$

and $$G_3(z^2) = \left(-\frac{1}{2}-\frac{\sqrt{14}}{16}\right) - \frac{7\sqrt{14}}{56}z^{-2} + \left(\frac{1}{2}-\frac{\sqrt{14}}{16}\right)z^{-4}$$

On solving we get the Laurent polynomial $\{H_1, H_2, H_3\}$ for the tight framelet generated from quadratic B-spline function B_3 as,

$$H_1 = -\frac{1}{128}e^{-4i\omega} - \frac{1}{32}e^{-3i\omega} - \frac{3}{32}e^{-2i\omega} - \frac{7}{32}e^{-i\omega} + \frac{45}{64} - \frac{7}{32}e^{i\omega} - \frac{3}{32}e^{2i\omega} - \frac{1}{32}e^{3i\omega} - \frac{1}{128}e^{4i\omega}$$

$$H_2 = -\frac{1}{32}e^{-2i\omega} - \frac{1}{8}e^{-i\omega} - \frac{7}{32} + \frac{3}{4}e^{i\omega} - \frac{7}{32}e^{2i\omega} - \frac{1}{8}e^{3i\omega} - \frac{1}{32}e^{4i\omega}$$

$$\begin{aligned} H_3 = {} & \left(\frac{\sqrt{2}}{32} + \frac{\sqrt{28}}{256}\right) + \left(\frac{\sqrt{2}}{8} + \frac{\sqrt{28}}{256}\right)e^{i\omega} \\ & + \left(\frac{7\sqrt{28}}{896} + \frac{3\sqrt{2}}{16} + \frac{6\sqrt{28}}{256}\right)e^{2i\omega} \\ & + \left(\frac{7\sqrt{28}}{224} + \frac{\sqrt{2}}{8} + \frac{\sqrt{28}}{64}\right)e^{3i\omega} \\ & + \left(\frac{\sqrt{28}}{128} + \frac{21\sqrt{28}}{448}\right)e^{4i\omega} \\ & - \left(\frac{\sqrt{2}}{8} - \frac{\sqrt{28}}{64} + \frac{7\sqrt{28}}{224}\right)e^{5i\omega} \\ & - \left(-\frac{6\sqrt{28}}{256} + \frac{3\sqrt{2}}{16} + \frac{7\sqrt{28}}{56}\right)e^{6i\omega} \\ & - \left(\frac{\sqrt{2}}{8} - \frac{\sqrt{28}}{64}\right)e^{7i\omega} - \left(\frac{\sqrt{2}}{32} - \frac{\sqrt{28}}{256}\right)e^{8i\omega} \end{aligned}$$

Therefore, the corresponding B-spline tight framelets $\{\psi_4^1, \psi_4^2, \psi_4^3\}$ are

$$\begin{aligned} \psi_4^1(x) = {} & -\frac{1}{128}B_4(2x+4) - \frac{1}{32}B_4(2x+3) \\ & - \frac{3}{32}B_4(2x+2) - \frac{7}{32}B_4(2x+1) \\ & + \frac{45}{64}B_4(2x) - \frac{7}{32}B_4(2x-1) \\ & - \frac{3}{32}B_4(2x-2) - \frac{1}{32}B_4(2x-3) \\ & - \frac{1}{128}B_4(2x-4) \end{aligned}$$

$$\begin{aligned} \psi_4^2(x) = {} & -\frac{1}{32}B_4(2x+2) - \frac{1}{8}B_4(2x+1) - \frac{7}{32}B_4(2x) \\ & + \frac{3}{4}B_4(2x-1) - \frac{7}{32}B_4(2x-2) \\ & - \frac{1}{8}B_4(2x-3) - \frac{1}{32}B_4(2x-4) \end{aligned}$$

$$\begin{aligned} \psi_4^3 = {} & \left(\frac{\sqrt{2}}{32} + \frac{\sqrt{28}}{256}\right)B_4(2x) + \left(\frac{\sqrt{2}}{8} + \frac{\sqrt{28}}{256}\right)B_4(2x-1) \\ & + \left(\frac{7\sqrt{28}}{896} + \frac{3\sqrt{2}}{16} + \frac{6\sqrt{28}}{256}\right)B_4(2x-2) \\ & + \left(\frac{7\sqrt{28}}{224} + \frac{\sqrt{2}}{8} + \frac{\sqrt{28}}{64}\right)B_4(2x-3) \\ & + \left(\frac{\sqrt{28}}{128} + \frac{21\sqrt{28}}{448}\right)B_4(2x-4) \\ & - \left(\frac{\sqrt{2}}{8} - \frac{\sqrt{28}}{64} + \frac{7\sqrt{28}}{224}\right)B_4(2x-5) \\ & - \left(-\frac{6\sqrt{28}}{256} + \frac{3\sqrt{2}}{16} + \frac{7\sqrt{28}}{56}\right)B_4(2x-6) \\ & - \left(\frac{\sqrt{2}}{8} - \frac{\sqrt{28}}{64}\right)B_4(2x-7) \\ & - \left(\frac{\sqrt{2}}{32} - \frac{\sqrt{28}}{256}\right)B_4(2x-8) \end{aligned}$$

Cubic (fourth order) B-spline B_3 and the generated compactly supported tight framelets $\{\psi_4^1, \psi_4^2, \psi_4^3\}$ are demonstrated in Figure.3

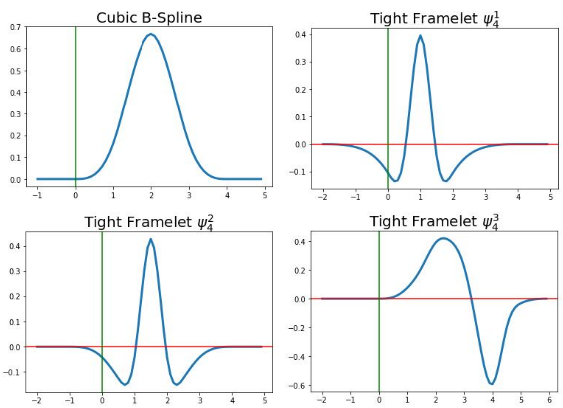

Figure 3: Cubic (fourth order) B-spline and generated tight framelets

5. CONCLUSION

Rudundancy is the main feature of frames. Due to this property, frame has found numerous real world application in field of mathematics and engineering. Also as splines have many exciting property and an explicit formulae, which make it the first choice to apply. Hence the spline wavelet frames has a very vast practical application in different fields. Here we had illustrate an effortless approach to construct compactly supported framelets with three generator associated through a polynomial refinable function. The B-spline framelets generated here are very similar to that obtained in [3,4,5].

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