ISSN 2278-3091
Volume 9 No.2, March - April 2020
International Journal of Advanced Trends in Computer Science and Engineering
Available Online at http://www.warse.org/IJATCSE/static/pdf/file/ijatcse218922020.pdf
https://doi.org/10.30534/ijatcse/2020/218922020

# The Use of Strong and Ordinal Scales during the Synthesis of Reference Images for Vehicle Correlation-Extreme Navigation Systems

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**ABSTRACT**

The results of the synthesis of the optimal reference image (RI) of the correlation-extreme navigation systems (CENS) of vehicles are presented in this article. They are represented in the ratio scales, differences scales and intervals scales that are relative to the strong scales as well as in the ordinal scales. The algorithm of the synthesis of the optimal RI in the ordinal scales is suggested.

**Key words:** Strong scales, ordinal scales, reference images, optimal algorithms, navigation systems.

## 1. INTRODUCTION

The solution of the problems of navigation of air-, sea- and land-launched vehicles with the use of autonomous correlation-extreme navigation systems is impossible without the corresponding reference images of the viewing surface.

The complexity of generation of the RI of the viewing surface is determined by the variation of conditions of formation of the original data, which is used for further RI synthesis. As a result of the lack of accurate data on the geometry of the viewing surface, weather conditions differences, the season when the original data has been gathered, weather conditions changes, the differences between the formed current image (CI) in the navigation system and the previously obtained RI can arise. Consequently, the accuracy characteristics of the navigation system can differ significantly from the required ones [1, 2, 3,4].

### 1.1 Problem analysis

One of the directions of synthesis of the optimal RI, which corresponds the fixed constituent of the informative parameter, is the use of the terms of scale of the Theory of Change. In the article [5] it has been suggested to choose the optimal RI, which corresponds the fixed constituent of the informative parameter, in each comparison act for the CI fragment. The description of the informative parameter is suggested to perform with the help of the term of the scale of the Theory of Change [6]. In accordance with this theory, the brightness component of RI, shifts, rotations of the RI with respect to the current image (CI), as well as an operation of the enumeration of CI fragments is suggested to formalize in terms of the Granander Image Theory, the fundamental principles of which are outlined in the article [7]. Therein, taking into consideration the necessity of provision of high precision navigation of vehicles, the most favorable option for the RI synthesis may be the representations in strong scales and the ordinal scales.

The goal of this article is the development of the RI synthesis method for the navigation systems of vehicles in the representations in strong scales and the ordinal scales.

## 2. MAIN MATERIAL

### 2.1The synthesis of the optimal RI, represented in the ratio scales, differences scales and intervals scales

The solution of the optimal RI synthesis problem in its representation in strong scales will be performed with the help of the Kuhn–Tucker theorem, which belongs to the finite-dimensional problems of the convex programming.

In accordance with this theorem, the statement [8] is valid:
1. let the functions$f$: $\mathbf{R}^N \to \mathbf{R}$, $g_1, \dots, g_m$: $\mathbf{R}^N \to \mathbf{R}$ be convex and continuously differentiable in$\mathbf{R}^N$;
2. let us assume that the vectors $\hat{\boldsymbol{\pi}} \in \mathbf{R}^N$and$\hat{\boldsymbol{\mu}} \in \mathbf{R}^m$satisfy the conditions:

$$\nabla_{\pi}(\hat{\pi},\hat{\mu}) = \nabla f(\hat{\pi}) + \nabla \mathbf{g}(\hat{\pi})\hat{\mu} = \mathbf{0}, \qquad (1)$$

$$\mathbf{g}(\hat{\pi}) \le \mathbf{0},\ \hat{\mu} \ge \mathbf{0},\ \hat{\mu}_j g_j(\hat{\pi}) = 0, \quad j \in \overline{1,m}. \qquad (2)$$

Then $\hat{\boldsymbol{\pi}}$ is the absolute minimum point of the problem:

$$\hat{\boldsymbol{\pi}} = \arg\min f(\boldsymbol{\pi}), \ \ \mathbf{g}(\boldsymbol{\pi}) \le \mathbf{0}. \tag{3}$$

In the representations (1)–(3):

$$L(\boldsymbol{\pi}, \boldsymbol{\mu}) = f(\boldsymbol{\pi}) + (\boldsymbol{\mu}, \mathbf{g}(\boldsymbol{\pi})) \tag{4}$$

the Lagrange functionof the problem (3), the vector inequality $\boldsymbol{\mu} \ge \mathbf{0}$ defines the inequality satisfaction for each vector component, and $\nabla \mathbf{g}(\boldsymbol{\pi})$ – $m \times N$ is the matrix with the elements $\frac{\partial g_i(\boldsymbol{\pi})}{\partial \pi_j}$, the columns of which are the gradients of function $g_i$.
Condition:

$$\hat{\mu}_j g_j(\hat{\boldsymbol{\pi}}) = 0, \ \ j \in \overline{1, m} \tag{5}$$

is the complementary slackness condition.
Let us consider the problem of the optimal RI synthesis:

$$\hat{g} = \arg \min_{g \in G} \|\mathbf{y} - g\mathbf{z}\|^2_{\mathbf{A}} \tag{6}$$

when it is represented in the ratio scales, differences scales and intervals scales that are relative to the strong scales.

**Ratio scale.** The group of the permissible transformations of the RI is represented as a one-parameter group of similarity transformation $g(\mathbf{x}) = \alpha\mathbf{x}, \alpha > 0$, the problem (6) is defined by the expression:

$$\hat{\alpha} = \arg\min_{\alpha \ge 0} f(\alpha) = \arg\min_{\alpha \ge 0} \|\mathbf{y} - \alpha\mathbf{z}\|^2_{\mathbf{A}}, \ \ \mathbf{y}, \mathbf{z} \in \mathbf{R}^M. \tag{7}$$

According to the comparison with the general case (1)-(3) it follows that $m = 1, g_1(\alpha) = -\alpha \le 0$. Functions $f(\alpha)$ and $g_1(\alpha)$ areconvex, as the first one is the quadratic function, while the second one is the linear function of the variable $\alpha$.
Let us form the Lagrange function of the problem (7):

$$L(\alpha, \mu) = \|\mathbf{y} - \alpha\mathbf{z}\|^2_{\mathbf{A}} - \alpha\mu, \tag{8}$$

for which the condition (1)is represented as:

$$-2(\mathbf{y}, \mathbf{z})_{\mathbf{A}} + 2\hat{\alpha}\|\mathbf{z}\|^2_{\mathbf{A}} - \hat{\mu} = 0, \tag{9}$$

and the complementary slackness condition (5) is represented as:

$$\hat{\mu}\hat{\alpha} = 0. \tag{10}$$

As $\hat{\alpha} \ge 0$ in accordance with (2) and $\|\mathbf{z}\|^2_{\mathbf{A}} \ge 0$, then $\hat{\alpha}\|\mathbf{z}\|^2_{\mathbf{A}} \ge 0$, and according to (10) $\hat{\mu}$ and $\hat{\alpha}$ can not be simultaneously distinct from zero, the solution of the equation (9) depends on the index of the scalar product $(\mathbf{y}, \mathbf{z})_{\mathbf{A}}$.
If $(\mathbf{y}, \mathbf{z})_{\mathbf{A}} > 0$, in accordance with $\mu \ge 0$ it follows that two cases are possible: $\hat{\mu} = \mu_1 = 0$ and $\hat{\mu} = \mu_2 \ne 0$.

In the first case, the equation (9) results in $\hat{\alpha} = (\mathbf{y}, \mathbf{z})_{\mathbf{A}} / \|\mathbf{z}\|^2_{\mathbf{A}} > 0$.
In the second case, the condition (10) results in $\hat{\alpha} = 0$ and according to the equation(10) it follows that $\hat{\mu} = -2(\mathbf{y}, \mathbf{z})_{\mathbf{A}} < 0$.
This condition contradicts the condition $\mu \ge 0$.
If $(\mathbf{y}, \mathbf{z})_{\mathbf{A}} \le 0$, then during the analysis of two cases, we find that the first case is contradictory, while the second one is equal to $\hat{\mu} = -2(\mathbf{y}, \mathbf{z})_{\mathbf{A}} \ge 0$ and $\hat{\alpha} = 0$.
Thus, the solution of the problem (7) is defined by the equation:

$$\hat{\alpha} = \begin{cases} (\mathbf{y}, \mathbf{z})_{\mathbf{A}} / \|\mathbf{z}\|^2_{\mathbf{A}}, & (\mathbf{y}, \mathbf{z})_{\mathbf{A}} > 0, \\ 0, & (\mathbf{y}, \mathbf{z})_{\mathbf{A}} \le 0, \end{cases} \tag{11}$$

and the lowest value of the decisive function is equal to:

$$f(\hat{\alpha}) = \|\mathbf{y}\|^2_{\mathbf{A}} - \begin{cases} (\mathbf{y}, \mathbf{z}')^2_{\mathbf{A}}, & (\mathbf{y}, \mathbf{z}')_{\mathbf{A}} > 0, \\ 0, & (\mathbf{y}, \mathbf{z}')_{\mathbf{A}} \le 0, \end{cases} \tag{12}$$

where $\mathbf{z}' = \mathbf{z} / \|\mathbf{z}\|$.

Thus, to ensure that the algorithm of the reference object localization is invariant to the scale distortion of the image, it is necessary to use the standardized RI and the decisive function (12).
**Difference scale.** The permissible transformation group (PTG) is the group of shifts in $\mathbf{R}$ ($g(\mathbf{x}) = \mathbf{x} + \beta, \beta \in \mathbf{R}$), the problem (6) can be determined by the expression:

$$\hat{\beta} = \arg\min f(\beta) = \arg\min \|\mathbf{y} - \mathbf{z} - \mathbf{1}_M \beta\|^2_{\mathbf{A}}. \tag{13}$$

where $\mathbf{1}_M$ is the vector in $\mathbf{R}^M$, all the components of which are equal to 1.
From (13) it follows that the problem is considered to finding the unconditional extremum of the function $f(\beta)$.
As a result of the differentiation, we'll obtain:

$$f'(\beta) = -2\bar{y} + 2\bar{z} + 2\beta; \ \ f''(\beta) = 2 > 0,$$

where $\bar{y} = \sum_{i=1}^{M} p_i y_i$, $\bar{z} = \sum_{i=1}^{M} p_i z_i$ are the weighted average values of the vectors $\mathbf{y}, \mathbf{z}$.
Therefore,the sought extremum is the minimum and the solution of the problem is defined with the expressions:

$$\hat{\beta} = \bar{y} - \bar{z}, \ \ f(\hat{\beta}) = \|\mathbf{y} - \mathbf{1}_M \bar{y} - (\mathbf{z} - \mathbf{1}_M \bar{z})\|^2_{\mathbf{A}}. \tag{14}$$

The vector $\mathbf{z} \in \mathbf{R}^M$ is centered to metric $\mathbf{A}$, if $(\mathbf{1}_M \mathbf{z})_{\mathbf{A}}$.
It is obvious that the vectors $\mathbf{y} - \mathbf{1}_M \bar{y}, \mathbf{z} - \mathbf{1}_M \bar{z}$ are $\mathbf{A}$-centered.
Thus, to ensure that the algorithm is invariant to the scale distortion of the current image or to the change of the brightness point of reference, it is necessary to center both RI and CI fragments while processing.

**Interval scale.** The PTG is represented as a linear group in $\mathbf{R}$ ($g(\mathbf{x}) = \alpha\mathbf{x} + \beta, \beta \in \mathbf{R}, \alpha > 0$), the problem is given by:

$$\left(\hat{\alpha},\hat{\beta}\right)=\arg\min_{\alpha\ge 0} f(\alpha,\beta)=\arg\min_{\alpha\ge 0}\|\mathbf{y}-\alpha\mathbf{z}-\beta\|_{\mathbf{A}}^{2}. \quad (15)$$

Since the unconstrained minimum is sought in accordance with the variable β, it can be given that the optimal value is
$\hat{\beta}=\bar{y}-\alpha\bar{z}$,using it in (15), results in the following problem:

$$\hat{\alpha}=\arg\min_{\alpha\ge 0} f(\alpha,\hat{\beta})=\arg\min_{\alpha\ge 0}\|\mathbf{y}'-\alpha\mathbf{z}'\|_{\mathbf{A}}^{2}, \quad (16)$$

where $\mathbf{y}'=\mathbf{y}-\mathbf{1}_M\bar{y},\ \ \mathbf{z}'=\mathbf{z}-\mathbf{1}_M\bar{z}$.

Problem (16) is solved while investigation of the case of RI representation in ratio scale, where it is shown that $\hat{\alpha}=(\mathbf{y}',\mathbf{z}'')_{\mathbf{A}}$:

$$f(\hat{\alpha},\hat{\beta})=\|\mathbf{y}'\|_{\mathbf{A}}^{2}-\begin{cases}(\mathbf{y}',\mathbf{z}'')_{\mathbf{A}}^{2}, & (\mathbf{y}',\mathbf{z}'')_{\mathbf{A}}>0,\\ 0, & (\mathbf{y}',\mathbf{z}'')_{\mathbf{A}}\le 0,\end{cases} \quad (17)$$

where$\mathbf{z}''=\mathbf{z}'/\|\mathbf{z}'\|$.
Hence, to ensure that the algorithm of RO localization is invariant to the group of linear transformations of the current image, it is necessary to use the centered CI fragments, as well as the centered and standardized RI.

### 2.2 The synthesis of the optimal reference image when represented in the ordinal scale

In this case $R$is the linear quasiorder on $X$.
The relationR is represented as the sum of the equivalence relation $R^I$and the strict order relation$R^S$, which establishes a linear order relation on a set of zones N. Assuming that the zones are renumbered in order of increasing brightness, the set of acceptable digitizations by zones can be described as follows:

$$C_{\boldsymbol{\pi}}=\left\{\boldsymbol{\pi}\in\mathbf{R}^N\middle|\pi_1<\cdots<\pi_N\right\}.$$

We take into account that the linear operator $h:\mathbf{R}^N\to\mathbf{R}^M$, having a matrix **H,**allows to build the digitization of the RI **κ**in accordance withthe vector of brightness of the zones digitization.
The matrix**H**has the elements$h_{ij}=\boldsymbol{\delta}_{r_i j}$ ,where$r_i$ is the number of the zone to which the i-th element of RI corresponds.

As a rule, the vector **y**,when compared with the digitization of RI, is pre-centered. The vector$\mathbf{y}\in\mathbf{R}^M$is called centered in the metric**A** or **A-**centered, if

$$(\mathbf{1}_M,\mathbf{y})_{\mathbf{A}}=\mathbf{1}_M'\mathbf{A}\mathbf{y}=\sum_{j=1}^{M}p_j y_j=0.$$

The following clauses are possible:
<u>Clause 1.</u>
If the vector**y** in the problem

$$\hat{\boldsymbol{\kappa}}=\arg\min_{\boldsymbol{\kappa}\in C_{\boldsymbol{\kappa}}\cap S_M}\|\mathbf{y}-\boldsymbol{\kappa}\|_{\mathbf{A}}^{2},$$

where$S_M=\{\mathbf{k}\in\mathbf{R}^M|\|\mathbf{k}\|_{\mathbf{A}}^2=\mathbf{1}\}$is a unit sphere in$\mathbf{R}^M$, is**A-**centered, then optimal digitization $\hat{\boldsymbol{\kappa}}$is equally**A-**centered.

<u>Proof.</u>From the chain of equalities

$$\mathbf{1}_M'\mathbf{A}\hat{\boldsymbol{\kappa}}=\sum_{i=1}^{M}p_i\hat{\kappa}_i=\sum_{i=1}^{M}p_i\hat{\pi}_{r_i}=\sum_{i=1}^{N}\hat{\pi}_i\sum_{k\in N_i}p_k=$$
$$=\sum_{i=1}^{N}\sum_{k\in N_i}p_k y_k=\sum_{i=1}^{M}p_k y_k=0$$

the proposition statement follows.
In the case of centered and normalized RI and CI fragments, the decisive function has the form:

$$B(k,\boldsymbol{\pi}^k)=2\left(1-\sum_{i=1}^{N}m_i^k\overline{y_i^k}e_i\right),$$

and the algorithm is equivalent to the correlation one.
<u>Clause2.</u>
Narrowing the image $h:\mathbf{R}^N\to\mathbf{R}^M$

$$h|_{C_{\boldsymbol{\pi}}}=C_{\boldsymbol{\pi}}\longrightarrow C_{\boldsymbol{\kappa}}=\{\boldsymbol{\kappa}\in\mathbf{R}^M|\boldsymbol{\kappa}=\mathbf{H}\boldsymbol{\pi},\boldsymbol{\pi}\in C_\pi\}$$

is bijective.

<u>Proof.</u>
It is known [9] that for an image to be bijective, it is necessary and sufficient that it is both injective and surjective.
<u>Injectivity.</u>Note that for every$\pi\in C_\pi$the way of reflection h is determined by a vector with components:

$$\kappa_i=\sum_{j=1}^{M}\delta_{r_i j}\pi_j=\pi_{r_i},\ \ i\in\overline{1,M}.$$

Let us take two arbitrary vectors$\boldsymbol{\pi}',\boldsymbol{\pi}''\in C_{\boldsymbol{\pi}},\ \boldsymbol{\pi}'\ne\boldsymbol{\pi}''$, find the corresponding images$\boldsymbol{\kappa}'=\mathbf{H}\boldsymbol{\pi}',\ \boldsymbol{\kappa}''=\mathbf{H}\boldsymbol{\pi}''$and suppose that $\boldsymbol{\kappa}'=\boldsymbol{\kappa}''$that is

$$\boldsymbol{\kappa}'-\boldsymbol{\kappa}''=\left(\pi_{r_1}'-\pi_{r_1}'',\dots,\pi_{r_M}'-\pi_{r_M}''\right)'=0.$$

This equality holds only for $\boldsymbol{\pi}'=\boldsymbol{\pi}''$, which proves the injectivity of the reflection$h|_{C_{\mathbf{k}}}$.
<u>Surjectivity</u> follows from the construction of the set$C_{\mathbf{k}}$.
<u>Definition.</u>The set$C\subset\mathbf{R}^N$is considered a cone if

$$\boldsymbol{\pi}\in C\Rightarrow\lambda\boldsymbol{\pi}\in X,\lambda>0.$$

<u>Definition.</u>The set$C\subset\mathbf{R}^N$is considered a convex if $\forall\pi',\pi''\in\mathbf{R}^N$the section connecting the points$\pi'$, $\pi''$belongs to C, that is.

$$\left\{\lambda\pi'+(1-\lambda)\pi'',\ \lambda\in[0,1]\right\}\subset C.$$

Clause3.
$C_{\boldsymbol{\pi}} = \{\boldsymbol{\pi} \in \mathbf{R}^N | \pi_1 < \cdots < \pi_N\}$ is a convex cone in $\mathbf{R}^N$.
Proof. If $\boldsymbol{\pi} \in C_\pi$, then $\lambda\boldsymbol{\pi} \in C_\pi$, as the system of inequalities $\lambda\pi_1 \leq \cdots \leq \lambda\pi_N$ is equivalent to the system $\pi_1 \leq \cdots \pi_N$, that is $C_\pi$ is a cone in $\mathbf{R}^N$.
To prove the convexity of the cone $C_{\boldsymbol{\pi}}$ let's take arbitrary vectors $\boldsymbol{\pi}', \boldsymbol{\pi}'' \in \mathbf{C}_{\boldsymbol{\pi}}$.
Multiplying the system of inequalities $\pi_1' \leq \cdots \leq \pi_N'$ by $\lambda \in [0,1]$, and the system $\pi_1'' \leq \cdots \leq \pi_N''$ by $1-\lambda$, and given that $\lambda, 1-\lambda \geq 0$, we will receive:

$$\lambda\pi_1' + (1-\lambda)\pi_1'' \leq \cdots \leq \lambda\pi_N' + (1-\lambda)\pi_N''.$$

Corollary. $C_{\boldsymbol{\kappa}} = \{\boldsymbol{\kappa} \in \mathbf{R}^M | \boldsymbol{\kappa} = \mathbf{H}\boldsymbol{\pi}, \boldsymbol{\pi} \in C_{\boldsymbol{\pi}}\}$ is a convex cone in $\mathbf{R}^M$.
Proof follows from the convexity of the cone $C_{\boldsymbol{\pi}}$ and from the bijectivity of the reflection: $h|_{C_{\boldsymbol{\pi}}} = C_{\boldsymbol{\pi}} \longrightarrow C_{\boldsymbol{\kappa}}, h(\boldsymbol{\pi}) = \mathbf{H}\boldsymbol{\pi}$.
The problem of finding the optimal digitization of RI has the form:

$$\hat{\boldsymbol{\kappa}} = \arg\min_{\boldsymbol{\kappa} \in C_{\boldsymbol{\kappa}}} \|\mathbf{y} - \boldsymbol{\kappa}\|_{\mathbf{A}}^2 \tag{18}$$

where

$$C_{\boldsymbol{\kappa}} = \left\{\boldsymbol{\kappa} \in \mathbf{R}^M \,\middle|\, \boldsymbol{\kappa} = \mathbf{H}\boldsymbol{\pi},\ \pi \in C_{\boldsymbol{\pi}} = \left\{\pi \in \mathbf{R}^N \middle| \pi_1 < \cdots < \pi_N\right\}\right\}.$$

Based on the Clause1this problem is equivalent to the following:

$$\hat{\boldsymbol{\pi}} = \arg\min f(\boldsymbol{\pi}) = \arg\min \|\mathbf{y} - \mathbf{H}\boldsymbol{\pi}\|_{\mathbf{A}}^2 \tag{19}$$

under restrictions:

$$g_j(\boldsymbol{\pi}) = \pi_j - \pi_{j+1} \leq 0,\ j \in \overline{1, N-1}. \tag{20}$$

By direct calculations, we find the vector

$$\mathbf{H}'\mathbf{A}\mathbf{y} = \left(\sum_{j \in N_1} p_j y_j, \ldots, \sum_{j \in N_N} p_j y_j\right)'$$

and matrix $\nabla\mathbf{g}(\pi)$, the columns of which are the gradients of function $g_j$:

$$\nabla\mathbf{g}(\pi) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Then the vector $\nabla\mathbf{g}(\pi)\mu \in \mathbf{R}^{N-1}$ is defined by the expression:

$$\nabla\mathbf{g}(\pi)\boldsymbol{\mu} = (\mu_1, \mu_2 - \mu_1, \ldots, \mu_{N-1} - \mu_{N-2}, -\mu_{N-1})'. \tag{21}$$

Let us write the system of equations for determining vectors $\hat{\boldsymbol{\pi}}, \hat{\boldsymbol{\mu}}$ in the coordinates taking into account (21), for simplicity omitting the tildes of the components of these vectors:

$$\begin{cases} 2n_1(\pi_1 - \pi_1') + \mu_1 = 0; \\ 2n_2(\pi_2 - \pi_2') + \mu_2 - \mu_1 = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2n_{N-1}(\pi_{N-1} - \pi_{N-1}') + \mu_{N-1} - \mu_{N-2} = 0; \\ 2n_N(\pi_N - \pi_N') - \mu_{N-1} = 0; \\ \mu_j(\pi_j - \pi_{j+1}) = 0,\ j \in \overline{1, N-1}, \end{cases} \tag{22}$$

where the vector $\boldsymbol{\pi}'$ is defined by the relation:

$$\hat{\pi}_i = \frac{1}{n_i} \sum_{k \in N_i} p_k y_k,\ i \in \overline{1,N}$$

which follows from the chain of equalities:

$$h_{ij} = \delta_{r_i j},\ i \in \overline{1,M},\ j \in \overline{1,N};$$

$$\mathbf{F} = \mathbf{H}'\mathbf{A},\ f_{ij} = \sum_{k=1}^{M} \delta_{r_k i} p_k \delta_{kj} = p_j \delta_{r_j i},\ i \in \overline{1,M},\ j \in \overline{1,N};$$

$$\mathbf{B} = \mathbf{F}\mathbf{H},\ b_{ij} = \sum_{k=1}^{M} \delta_{r_k i} p_k \delta_{r_k j} = \delta_{ij} n_i,\ i,j \in \overline{1,M};$$

$$\mathbf{C} = \mathbf{B}^{-1},\ c_{ij} = \delta_{ij}\alpha_i,\ \alpha_i = 1/n_i,\ i,j \in \overline{1,M};$$

$$\mathbf{D} = \mathbf{C}\mathbf{H}',\ d_{ij} = \sum_{k=1}^{M} \delta_{r_j k} \alpha_i \delta_{ik} = \alpha_i \delta_{r_i j},\ i \in \overline{1,M},\ j \in \overline{1,N};$$

$$\mathbf{E} = \mathbf{D}\mathbf{A},\ e_{ij} = \sum_{k=1}^{M} \alpha_i \delta_{r_k i} p_k \delta_{kj} = \alpha_i \delta_{r_j i} p_j,\ i \in \overline{1,M},\ j \in \overline{1,N}.$$

To solve system (22), we will use the method of successive exclusion of variables, starting with the first equation. In the absence of restrictions (when $\boldsymbol{\mu} = \mathbf{0}$) we receive $\boldsymbol{\pi} = \boldsymbol{\pi}'$. Therefore, in the first step, we check the order relation using the elements $\pi_1', \pi_2'$. If $\pi_1' < \pi_2'$ then in accordance to the complementary slackness condition (6) it follows that $\mu_1 = 0, \pi_1 = \pi_2$ and the first two equations are excluded from the system (22).
If during the i-th step it happened that $\pi_i' \geq \pi_{i+1}'$ Let's call this situation a block of inversions. To satisfy the condition $\pi_i \leq \pi_{i+1}$ it is necessary to put $\pi_i = \pi_{i+1}$ and $\mu_i > 0$ up to this step we have $\mu_1 = \cdots = \mu_{i-1} = 0$, $\pi_k = \pi_k' (k \in \overline{1, i-1})$, and the system (22) takes the form of:

$$\begin{cases} 2n_i(\pi_i - \pi_i') + \mu_i = 0; \\ 2n_{i+1}(\pi_{i+1} - \pi_{i+1}') + \mu_{i+1} - \mu_i = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2n_{N-1}(\pi_{N-1} - \pi_{N-1}') + \mu_{N-1} - \mu_{N-2} = 0; \\ 2n_N(\pi_N - \pi_N') - \mu_{N-1} = 0; \\ \mu_j(\pi_j - \pi_{j+1}) = 0,\ j \in \overline{i, N-1}. \end{cases} \tag{23}$$

Using the first equation of system (23) we find $\mu_i = 2n_i(\pi_i' - \pi_{i+1})$, while adding up the first two equations, we get:

$$2n_{i+1}'(\pi_{i+1} - \pi_{i+1}'') + \mu_{i+1} = 0,$$

where

$$n'_{i+1} = n_i + n_{i+1},\ \pi''_{i+1} = \left(n_i\pi'_i + n_{i+1}\pi'_{i+1}\right)/n'_{i+1}.$$

Thus, in this step, the following variables are excluded: $\mu_i, \pi_i$.Further, if$\pi''_{i+1} \geq \pi'_{i+2}$,then we assume $\pi_{i+1} \geq \pi_{i+2}$and find$\mu_{i+1} = 2n'_{i+1}(\pi''_{i+1} - \pi_{i+2})$.
Let on the$(i+1)$-th step the condition$\pi''_{i+1} < \pi'_{i+l+1}$ is satisfied, which corresponds to the end of the block of inversions. The number l is called the length of the block of inversions. To this step, system (22) takes the form:

$$\begin{cases} 2n'_{i+l}\left(\pi_{i+l} - \pi''_{i+l}\right) + \mu_{i+l} = 0; \\ 2n_{i+l+1}\left(\pi_{i+l+1} - \pi'_{i+l+1}\right) + \mu_{i+l+1} - \mu_{i+l} = 0; \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 2n_{N-1}\left(\pi_{N-1} - \pi'_{N-1}\right) + \mu_{N-1} - \mu_{N-2} = 0; \\ 2n_N\left(\pi_N - \pi'_N\right) - \mu_{N-1} = 0; \\ \mu_j\left(\pi_j - \pi_{j+1}\right) = 0,\ j \in \overline{1, N-1}. \end{cases} \tag{24}$$

Then$\mu_{i+1} = 0$and from the first equation of system (24) we find:

$$\pi_{i+1} = \dots = \pi_{i+l} = \pi''_{i+1}, \tag{25}$$

where

$$\pi''_{i+k} = \sum_{j=i}^{i+k} n_j\pi'_j / n'_{i+k},\ k \in \overline{0, l-1}, \tag{26}$$

$$n'_{i+k} = \sum_{j=i}^{i+k} n_j. \tag{27}$$

Furthermore,

$$\mu_{i+k} = 2\left(\sum_{j=i}^{i+k} n_j\pi'_j - \pi_{i+l}\sum_{j=i}^{i+k} n_j\right),\ k \in \overline{0, l-1} \tag{28}$$

Using (26), (27), expression (28) can be transformed to:

$$\mu_{i+k} = \frac{2}{n'_{i+l}} \sum_{p=i}^{i+k} \sum_{j=i+k+1}^{i+l} n_p n_j\left(\pi'_p - \pi'_j\right),\ k \in \overline{0, l-1}. \tag{29}$$

Continuing in a similar way the process of solving the system (22) with the allocation of blocks of inversions, we can solve all the equations and find the vector $\boldsymbol{\pi}$. In order for this vector to be optimal, i.e. to be a solution to problem (19), (20), it remains to demonstrate that $\mu_{i+k} \geq 0,\ k \in \overline{0, l-1}$.

If in the process of solving system (22) it turns out that the block of inversions begins with the first equation and ends with the last, then, as follows from formula (25) while $i = 1, l = N-1$, the solution is defined by the vector:

$$\pi_1 = \dots = \pi_N = \sum_{k=1}^{M} p_k y_k.$$

Let us rewrite the expression (28) and the system of inequalities that are satisfied at each step of the block of inversion, in the form:

$$\mu_{i+k} = \frac{2}{\sum_{j=i}^{i+l} n_j} \sum_{j=i+k+1}^{i+l} n_j\left(\sum_{p=i}^{i+k} n_p\pi'_p - \pi'_j\sum_{p=i}^{i+k} n_p\right), \tag{30}$$

$$\sum_{p=i}^{i+k} n_p\pi'_p \geq \pi'_{i+k+1}\sum_{p=i}^{i+k} n_p,\ k \in \overline{0, l-1}. \tag{31}$$

Lemma 1.$\mu_{i+k} \geq 0, k \in \overline{0, l-1}$.
Proof.Since$n_j > 0$, it suffices to prove that all terms of the inner sum in (30) are non-negative, i.e.:

$$\sum_{p=i}^{i+k} n_p\pi'_p \geq \pi'_j\sum_{p=i}^{i+k} n_p,\ j \in \overline{i+k+1, i+l-1},\ k \in \overline{0, l-1}. \tag{32}$$

To prove (32), we use the method of mathematical induction on j.
At$j = i + k + 1$the system (32) consists of one inequality, which coincides with inequality (31).
Suppose that the statement of the lemma is true for$j = i + l - 2$, i.e.

$$\sum_{p=i}^{i+k} n_p\pi'_p \geq \pi'_j\sum_{p=i}^{i+k} n_p,\ j \in \overline{i+k+1, i+l-2}, k \in \overline{0, l-1} \tag{33}$$

Let us prove that it is valid for$j = i + l - 1$.
We write inequality (31) for $k = l - 1$:

$$\sum_{p=i}^{i+k} n_p\pi'_p + \sum_{j=i+k+1}^{i+l-1} n_j\pi'_j \geq \pi'_{i+l}\sum_{p=i}^{i+l-1} n_p$$

and strengthen it by replacing $\pi'_j$ by $\sum_{p=i}^{i+k} n_p\pi'_p \Big/ \sum_{p=i}^{i+k} n_p,\ \left(j \in \overline{i+k+1, i+l-2}\right)$, which is admissible by virtue of the system of inequalities (33).

As a result, we obtain:

$$\sum_{p=i}^{i+k} n_p\pi'_p\left(1 + \sum_{p=i+k+1}^{i+l-1} n_p \Big/ \sum_{p=i}^{i+k} n_p\right) \geq \pi'_{i+l}\sum_{p=i}^{i+l-1} n_p,$$

where, given the fact that $n_j > 0\ (j \in \overline{1, N})$, the inequality follows:

$$\sum_{p=i}^{i+k} n_p\pi'_p \geq \pi'_{i+l}\sum_{p=i}^{i+k} n_p,\ k \in \overline{0, l-1}. \tag{34}$$

Adding inequality (34) to system (33), we obtain statement (32), which completes the proof of the lemma.

Clause4.
If the CI vector$\mathbf{y} \in \mathbf{R}^M$is centered in the metric $\mathbf{A}$, then the vector$\boldsymbol{\kappa} = \mathbf{H}\boldsymbol{\pi}$, where$\boldsymbol{\pi}$is the solution of the problem (19), (20), is$\mathbf{A}$-centered.

Proof.

The following chain of equalities holds:

$$(\mathbf{1}_M,\boldsymbol{\kappa})_{\mathbf{A}}=\sum_{j=1}^{M}p_j\kappa_j=\sum_{i=1}^{M}p_i\sum_{j=1}^{N}h_{ij}\pi_j=\sum_{j=1}^{N}\pi_j\sum_{i=1}^{M}\delta_{r_ij}\pi_j=\sum_{j=1}^{N}\pi_j\sum_{k\in N_j}p_k=\sum_{j=1}^{N}\pi_j n_j=$$
$$=\sum_{j\in Q_1}\pi_j n_j-\sum_{j\in Q_2}\pi_j n_j, \quad (35)$$

where $n_j$ is defined by the expression $n_i=\sum_{k\in N_i}p_k$;

$Q_1\subset\overline{1,N}$ is the set of components of the vector $\boldsymbol{\pi}$, not falling into blocks of inversions;

$Q_2=\overline{1,N}\setminus Q_1$.

Let J is the number of blocks of inversions. Then the set $Q_2$ can be represented as

$$Q_2=\bigcup_{q=1}^{J}\overline{i_q,i_q+l_q}$$

Where $i_q$ is the number of the vector component $\boldsymbol{\pi}$, c which is the beginning point of the q-thblock of inversions;

$l_q$ is its length.

Then the relation (35) can be written as follows:

$$(\mathbf{1}_M,\boldsymbol{\kappa})_{\mathbf{A}}=\sum_{j\in Q_1}\pi_j n_j+\sum_{q=1}^{J}\sum_{j=i_q}^{i_q+l_q}\pi_j n_j. \quad (36)$$

From the expression (25) and the procedure for solving the system (22) it follows:

$$\pi_j=\pi_j'=\frac{1}{n_j}\sum_{k\in N_j}\pi_k y_k,\quad j\in Q_1; \quad (37)$$

$$\pi_j=\dots=\pi_{j+1}=\sum_{p=i_q}^{i_q+l_q}n_p\pi_p'\Big/\sum_{p=i_q}^{i_q+l_q}n_p=\sum_{p=i_q}^{i_q+l_q}\sum_{k\in n_p}\pi_k y_k\Big/\sum_{p=i_q}^{i_q+l_q}n_p, \quad (38)$$
$$j\in\overline{i_q,l_q},\ q\in\overline{1,J}$$

Substituting (37), (38) in (36), we obtain:

$$(\mathbf{1}_M,\boldsymbol{\kappa})_A=\sum_{j\in Q_1}\sum_{k\in N_j}\pi_k y_k+\sum_{q=1}^{J}\sum_{j=i_q}^{i_q+l_q}\sum_{k\in N_j}\pi_k y_k=$$
$$=\sum_{j\in Q_1}\sum_{k\in N_j}\pi_k y_k+\sum_{j\in Q_2}\sum_{k\in N_j}\pi_k y_k=$$
$$=\sum_{j=1}^{N}\sum_{k\in N_j}\pi_k y_k=\sum_{k=1}^{M}\pi_k y_k=0,$$

since the vector $\mathbf{y}$ is $\mathbf{A}$-centered.

It is of interest to obtain a normalized solution to problem (18), i.e. to solve a problem:

$$\hat{\kappa}=\arg\min_{\kappa\in C_\kappa\cap S_M}\|\mathbf{y}-\kappa\|_{\mathbf{A}}^2, \quad (39)$$

which is equivalent to the problem of minimizing:

$$f(\pi)=\|\mathbf{y}-\mathbf{H}\pi\|_{\mathbf{A}}^2 \quad (40)$$

under restrictions

$$g(\boldsymbol{\pi})=\pi_i-\pi_{i+1}\le 0,\ i\in\overline{1,N-1},\quad \nu(\boldsymbol{\pi})=\|\mathbf{H}\pi\|_{\mathbf{A}}^2-1=0. \quad (41)$$

Clause5.

To obtain a solution to problem (40), (41), it is necessary to solve problem (19), (20) and normalize the resulting solution.

Proof.

The Lagrangian of problem (40), (41) is determined by the expression:

$L(\boldsymbol{\pi},\lambda,\boldsymbol{\mu})=f(\boldsymbol{\pi})+\lambda\nu(\boldsymbol{\pi})+\boldsymbol{\mu}'g(\boldsymbol{\pi})$.

System of equations for determining the optimal vector $(\boldsymbol{\pi},\lambda,\boldsymbol{\mu})\in\mathbf{R}^{2N}$ has the form:

$$2[\mathbf{H}'\mathbf{A}\mathbf{H}(1+\lambda)-\mathbf{H}'\mathbf{A}\mathbf{y}]+\nabla g\boldsymbol{\mu}=\mathbf{0};$$
$$\mu_j(\pi_j-\pi_{j+1})=0,\ j\in\overline{1,N-1};$$
$$\boldsymbol{\pi}'\mathbf{H}'\mathbf{A}\mathbf{H}\boldsymbol{\pi}=1,$$

or in the coordinates

$$\begin{cases}2n_1[\pi_1(1+\lambda)-\pi_1']+\mu_1=0;\\ 2n_2[\pi_2(1+\lambda)-\pi_2']+\mu_2-\mu_1=0;\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ 2n_{N-1}[\pi_{N-1}(1+\lambda)-\pi_{N-1}']+\mu_{N-1}-\mu_{N-2}=0;\\ 2n_N[\pi_N(1+\lambda)-\pi_N']-\mu_{N-1}=0;\\ \mu_j(\pi_j-\pi_{j+1})=0,\ j\in\overline{1,N-1};\\ \sum_{i=1}^{N}p_i\pi_i^2-1=0.\end{cases} \quad (42)$$

Comparing the system of equations (42) with the system (22), we conclude that it is possible to solve the first 2m-1 equations by the previously described method and to find the vector $\boldsymbol{\pi}''=\boldsymbol{\pi}(\lambda+1)$ and then to determine $\lambda$ from the last equation:

$$\|\boldsymbol{\pi}''\|_{\mathbf{A}}^2=(1+\lambda)^2,\quad \lambda=\|\boldsymbol{\pi}''\|_{\mathbf{A}}-1.$$

Hence, $\boldsymbol{\pi}=\boldsymbol{\pi}''/\|\boldsymbol{\pi}''\|_{\mathbf{A}}$.

The following algorithm 1 follows from the methodology for solving system (22) and Clauses 1 and 2.

Step 1. Set the vectors $\mathbf{p}=(p_1,\dots,p_M)$, $p_i>0$, $\sum_{i=1}^{M}p_i=1$; $\mathbf{y}\in\mathbf{R}^M$; $\mathbf{r}\in\overline{1,N}^M$ where the component $r_i$ of the vector $\mathbf{r}$ is equal to the number of the zone to which the $i$-thelement of RI belongs.

Step 2. Build the matrix $\mathbf{H}$ with elements $h_{ij}=\delta_{r_ij}$, where $\delta_{ij}$ is the Kronecker symbol, $i\in\overline{1,M}$, $j\in\overline{1,N}$.

Step 3. Build the set $N_i=\{j\in\overline{1,M}\,|\,r_j=i\}$, $i\in\overline{1,N}$

Step 4. Build the vectors $\mathbf{n}\in\mathbf{R}^N$, $n_i=\sum_{k\in N_i}p_k$, $\boldsymbol{\pi}'\in\mathbf{R}^N$

$$\pi_i'=\frac{1}{n_i}\sum_{k\in N_i}p_k z_k.$$

Step 5. Set $i=1$.

Step 6. Set $j=i$, $l=1$, $s_1=\pi_i'n_i$, $s_2=n_i$

Step 7. If $\pi'_j > \pi'_{j+1}$ and $j \le N-1$ then repeat $l := l+1$, $s_1 := s_1 + n_{j+1}\pi'_{j+1}$, $s_2 := s_2 + n_{j+1}$, $\pi'_k = s_1/s_2$, $k \in \overline{j-1+2, j+1}$, $j := j+1$.

Step 8. If $l = 1$ then continue $i := i+1$ or $i = j$.

Step 9. If $i \le N-1$ then go to step 6.

Step 10. Construct an optimal digitization vector $\hat{\boldsymbol{\kappa}} = \mathbf{H}\boldsymbol{\pi}'$.

Theorem1.
The vector $\hat{\boldsymbol{\kappa}} = \mathbf{R}^M$, constructed by Algorithm 1 is a solution of problem (39) centered and normalized in the metric A.

Proof.
Problem (39) is equivalent to problem (40), (41), and the conditions of the Kuhn-Tucker theorem for the latter have the form:

$$\nabla_{\pi} L(\boldsymbol{\pi}, \lambda, \boldsymbol{\mu}) = \mathbf{0}; \tag{43}$$

$$\mu_j(\pi_j - \pi_{j+1}) = 0, \ \ j \in \overline{1, N-1}; \tag{44}$$

$$\|\mathbf{H}\boldsymbol{\pi}\|^2_{\mathbf{A}} - 1 = 0; \tag{45}$$

$$\mu_j \ge 0, \ j \in \overline{1, N-1}\,. \tag{46}$$

Using Algorithm 1, the vector $\boldsymbol{\pi}''$ is first calculated in accordance with the methodology for solving the system of equations (22), therefore, the vector $\boldsymbol{\pi}''$ along with the vector $\boldsymbol{\mu}$ is a solution to system (22), i.e. conditions (43), (44) are satisfied.
It follows from Clause 5 that the vector $(\boldsymbol{\pi}, \lambda, \boldsymbol{\mu}) \in \mathbf{R}^{2N}$ is a solution to the system of equations (42), i.e. condition (45) is also satisfied.
Condition (46) follows from Lemma 1, and the statement that the vector is centered $\boldsymbol{\kappa} = \mathbf{H}\boldsymbol{\pi}$- from the Clause 4.
Finally, the statement of the theorem follows from the equivalence of problem (40), (41) to problem (39).

## 3. CONCLUSION

After the research conducted, the optimal reference image (RI) of the correlation-extreme navigation systems (CENS) of vehicles has been obtained, which is represented in the ratio scales, differences scales and intervals scales that are relative to the strong scales as well as in the ordinal scales.
The algorithm of the synthesis of the optimal RI in the ordinal scales has been developed.

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