ISSN 2278-3091
Volume 8, No.1.1, 2019
International Journal of Advanced Trends in Computer Science and Engineering
Available Online at http://www.warse.org/IJATCSE/static/pdf/file/ijatcse1481.12019.pdf
https://doi.org/10.30534/ijatcse/2019/1481.12019

Mergers of Operators and Regulation: a Game – Theoretic Approach

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ABSTRACT
Game theory allows modeling mergers, using two models, Cournot competition and Bertrand competition. One can study the behavior of the regulator. In case of Cournot competition he is secure if there are at least three large operators. In case of Bertrand competition, he fears an increase of the prices if a merger which is profitable occurs.

Keywords: Mergers, Regulation, Game Theory, Industrial Organization.

1. INTRODUCTION.

Game theory is not useful to predict the events accurately [1]. But it "gives ideas" or allows experiments of thought [2]. In particular, game theory allows avoiding an error: to neglect the strategic effect. The strategic effect is when a player having changed his choice, the other players have to change their choices, also. The theorist has to take into account the two effects, the direct effect and the strategic effect. It is obvious in the case of mergers. If one takes into account the direct effect only, all he mergers should be profitable: the merged firm could replicate the choices of the merging firms, and then optimize its gain, obtaining more (than the joint profit of the merging firms). But the strategic effect matters. And mergers can be unprofitable. More, as it is set out in the paper, there are three kinds of mergers (this applies to Cournot competition): (1) some are easy because they are profitable (2) some are difficult because they require "side payments" (3) some are desperate, because they are not profitable and do not trigger an increase of the total profit (therefore the "side payments" are not a recourse).

Game theory allows modeling competition between firms in two ways:

- Cournot competition. The firms choose the quantity they sell. Notating a firm E_i, its cost c_i, the quantity sold q_i and the price p (q) (q being the total quantity sold) and P_i the profit:
  $P_i = (p\,(q_1 + q_2 + q_3) - c_i)\, q_i$ (if there are n = 3 firms). And Nash equilibrium is given by:
  $\partial P_i / \partial q_i = 0$, i = 1, 2, 3.
  We make only the hypothesis that p (q) is concave and that the equilibriums exist and are unique. How to study the profitability of mergers?
  We consider that the merged firm has a cost which is the lower of the two costs of the merging firms. Therefore the criterion is: does the joint profit increase, when the merger occurs, the merged firm having the cost c_1 ($c_1 < c_2$)?
  The criterion is written
  $P > P_1 + P_2$ (1)
  This condition allows the merger since there is some price π:
  $P - \pi > P_1$ (the buyer makes a gain)
  $\pi - P_2 > 0$ (the seller makes a gain).
- Bertrand competition. The firms sell products which are partially substitutable and choose their price:
  $P_i = (p_i - c_i)\, D_i\, (p_1, p_2, p_3)$ (if n = 3)
  The Nash equilibrium is given by:
  $\partial P_i / \partial p_i = 0$, i = 1, 2, 3.
  The only hypothesis we make is that when one of the prices is fixed, the two other varying, the demand is concave. Also, we suppose the existence and unicity of the equilibriums.

What results are obtained thanks to game theory? What are the consequences for the regulator?
To answer this question one has to discriminate the two cases, Cournot competition and Bertrand competition:

Cournot competition.

One demonstrates that two firms with the same cost (or costs with close values) cannot merge because it is unprofitable. But if the spread between the two costs of the merging firms is large enough, the merger becomes profitable. In familiar words: two large operators cannot merge (except if there are only two firms in the sector) but a large one can buy a small one. Notice that it does not change the market very much. The regulator is "secure" as soon as there are at least three large, stable operators. Perhaps a merger of two large operators is profitable thanks to "strategic reasons" like a project involving synergies and economies of scale. But this merger is not feared by the regulator: in principle it does not trigger a rise of the price after the merger which would be detrimental to the consumers. Notice that the regulator remains useful: he has to appreciate the "strategic reasons" and he is indispensable if there are only two large operators in the sector (since in this case the merger is always profitable).

Sometimes but not always a merger which is not profitable could succeed thanks to side payments. We can give an example.

Suppose the gains of three operators E_1, E_2, E_3 are 50 for each before the merger and 80 for each after the merger. The merger is not profitable: the gain of the merged firm is less than the joint profit of the merging firms ($80 < 50 + 50 = 100$). But if there are side payments, the firm E_3 can give 25 to the merged firm E (after the merger of E_1 and E_2, there is E).

The firm E_3 wins more ($80 - 25 = 55 > 50$) and the gain of the merged firm E is more than the joint profit of the merging firms (80 + 25 = 105 > 50 + 50 = 100). Therefore the merger is profitable. It is possible, thanks to side payments.

But the negotiation between all the firms of the sector and the regulator which concerns the side payments is too complex. This merger should be difficult. [1] Again, the regulator is "secure".

And finally there are "desperate" cases: when a merger does not trigger an increase of the total profit, even "side payments" cannot allow the success of this merger. This exists. One will give an example.

Bertrand competition.

One demonstrates that if merger is chosen (because it is profitable) the three prices increase. This is detrimental to the consumers. The regulator fears this kind of situation, but intervention is uneasy except when two large firms (with big market shares) are concerned. It is because there is no clear criterion on the relevant market: are the two firms really in competition? Or do they sell differentiated products to different customers?

Our conclusion is that the regulator is more secure in case of Cournot competition, when a homogeneous product is sold, like mineral water, ore or staples …In the case of telecommunications services it could be the sale of traffic. When the products sold are differentiated, profitable mergers are possible, triggering a rise of the prices, and the regulator is in an awkward role.

This is confirmed if one considers that there are two possibilities, "buy and manage" (the purchased firm is managed) and "buy and close down" (the purchased firm is closed down). We shall show that "buy and close down" is sometimes profitable, thanks to examples. And the regulator can uneasily oppose a "buy and close down" which is detrimental to the consumers with preference for the product sold by the closed down firm. Indeed, the buyer can evoke the difficulty to manage the purchased firm after some time, and claim that he has to close down it.

Notice that the idea of unprofitable mergers in case of Cournot competition has been set out in papers in the 80's [3]. The framework of this paper is different. One studies mergers, not coalitions. One supposes that the cost of the merged firm is the smaller of the two costs of the merging firms. In general mergers involve two firms (at the opposite coalitions can involve more than two firms). In this paper one studies only this situation involving more than two firms: the "side payments". When a merger is not profitable but triggers more total profit it can succeed thanks to "side payments". All the firms of the sector participate in the negotiation.

The same remark holds concerning papers on Bertrand equilibrium [4]. In the quoted article the symmetrical case (the demands are symmetrical and the costs are equal) is studied. And the demands are supposed linear (or they are not linear but there are several conditions). At the opposite in this paper only the two merging firms are supposed symmetrical (symmetrical demands and equal costs). And any demand, not necessarily linear, is considered.

In the existing literature coalitions are more often studied than mergers and the costs are generally supposed equal. At the opposite, we consider that the cost of the merged firm is the lower of the two costs of the merging firms. The merger changes the way in which the merged firm is managed. The efficient methods used in the buying firm are applied in the purchased firm (for instance, upgrades in the network). It seems justified but it is not taken into account in general. Even the famous Cournot in his seminal work, considers that in the coalition of all the firms of a sector, the assets are not changed, but used in a different way: the more efficient plant will produce more, the less efficient plant will produce less [5]. Our method allows finding results which are in accordance with intuition: "peer operators", with the same cost, cannot merge because the merger does not allow more efficiency. And when the spread between the two costs (of the merging firms) is enough, the merger becomes profitable since the purchased firm benefits from the lower cost of the buying firm. Another difference with existing literature is that we make no particular hypothesis on the demands, except the concavity and the existence (and unicity) of the equilibrium.

Also, the topic of "side payments" seems to not have been dealt with in the literature. Finally, concerning Bertrand competition, we prefer to deal with demands which are deduced from the utilities. This is explained in the chapter 6 of the book "Game theory and the stakes in the telecommunications industry", written by the author of the paper [1]. To deduce the demands from the utilities allows demonstrating some inequalities (concerning the derived functions of the demand) which are useful.

Now we can present the plan of the paper. We start by the Cournot competition, giving examples, and then demonstrating that (1) two peer operators cannot merge and (2) when the spread between the costs of the merging firms is large enough, they can merge. Then we conclude on Cournot competition and the behavior of the regulator. Then the topic of the stability of the equilibriums is briefly treated. And we deal with the topic of Bertrand competition. An example of "buy and close down" is described. Then we conclude on the topic of market consolidation and the behavior of the regulator.

2. CONSOLIDATION AND COURNOT COMPETITION (1): EXAMPLES.

We give one example of each kind of merger, easy, difficult and "desperate:

Example 1: an easy merger.

Let us define: the demand is $p=1-q$, $c_1=0$, $c_3=0$ and $c_2=1/4$.
With three competitors, $p=5/16$, $P_1=P_3= (5/16)^2$, $P_2= (1/16)^2$.
After the merger of E_1 and E_2, $p=1/3$, $P_1=P_3=1/9$.The condition (1) is fulfilled: $1/9 > (5/16)^2 + (1/16)^2$. The spread between the two costs $c_1=0$ and $c_2 = ¼$ is large enough and the merger is profitable.

Example 2: a difficult merger.

It is dealt with in the quoted book: a merger when all the costs are equal is always unprofitable, except when there only two competitors [1]. And the total profit increases (because the price increases towards the monopoly price when the number of firms passes from n to n-1). Therefore such a merger could succeed thanks to "side payments".

Example 3: a "desperate" case.

Suppose $c_1=c_2=0$, $c_3=1/3$. The merger considered is that of E_1 and E_2. Before the merger E_3 has a market share equal to 0. The total profit is 2/9. After the merger E_3 has a market share which is positive and the total profit is 17/81. The total profit

[1] In practice, the side payments have the shape of assets which are sold at a high price. The buyer accepts because he is interested in the merger, which triggers more profit for him. The third firm (if there are three firms) has an increasing profit because of the merger, since there is less competition.

has decreased: 17/81<2/9. It is counterintuitive. It shows that such a merger cannot succeed thanks to "side payments".

3. CONSOLIDATION AND COURNOT COMPETITION (2): THE MERGER OF TWO PEER OPERATORS IS NOT PROFITABLE.

There are two proofs. The first is short, intuitive and geometrical; the second is longer but with more intellectual rigor.

*Proof 1.*One considers the Figure 1 where there are in axes Oq_3q_2 (1) the reaction function R_3 of E_3 (2) the reaction function R of E_1 and E_2 of cost c when there is three competitors ($q_2 = 2q_0$) cutting R_3 in E (3) the reaction function R' of the merged firms cutting R_3 in E'. One proves that the condition for E is that the contour line of the joint profit in E is tangent to R_3. The contour lines are concave. Therefore there is a point on R', at the left of E', L, from which starts the contour line tangent to R_3 in E. Also, R_3 cannot cut the contour line starting from L because of the unicity of the equilibrium. The contour line passing by E' is on the right of the contour line passing by E.(Notice that R_3 could be the reaction function of n-2 firms, n > 3).

*Proof 2.*If $c>c_3$ the demonstration is in the chapter 3 of the quoted book. If $c<c_3$ one replaces the demand by two straight lines D_1 and D_2 tangent to the demand at the points corresponding to the equilibrium with three firms (D_1) and two firms (D_2). Then one imagines a straight line representing a demand rotating around the point of intersection of D_1 and D_2. The joint profit varies from less than this profit at the equilibrium with three firms to this profit. One proves that during the move it increases, or decreases then increases. And it passes by the value corresponding to D_2. So the joint profit corresponding to D_2 (equilibrium with two firms) is less than the joint profit corresponding to D_1 (equilibrium with three firms). One uses the formulas valid when the demand is linear, which are in the chapter 3 of the quoted book.

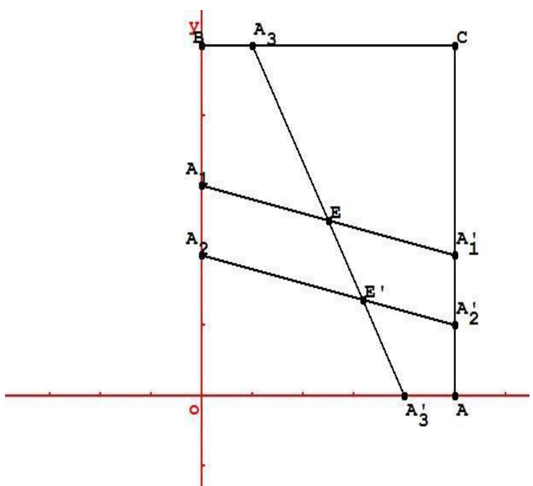

Figure 1: The replacement of the reaction function R by R' is shown. $A_3A'_3$: R_3, $A_1A'_1$: R, $A_2A'_2$: R'.

4. CONSOLIDATION AND COURNOT COMPETITION (3): IF THE SPREAD BETWEEN THE COSTS OF THE MERGING FIRMS IS LARGE ENOUGH THE MERGER BECOMES PROFITABLE.

There is two proofs, the first analytical the second geometrical.

Proof 1. One differentiates the equations describing the Nash equilibrium with three firms to obtain the changes in the values of the quantities when there is a little increase of c_2, $\Delta c_2>0$. One obtains: $\Delta q_1>0$, $\Delta q_2<0$, $\Delta q_3>0$, $\Delta q_1+\Delta q_2+\Delta q_3<0$, $\Delta p>0$, $\Delta P_1>0$, $\Delta P_2<0$, $\Delta P_3>0$. It is in accordance with intuition. Concerning the joint profit:

$$P_1 + \quad P_2 / D = p'^2 \Delta c_2 [2p' (q_1-q_2) + p'' (q_1^2 - q_1q_2 - q_2q_3)]. \quad (2)$$

If q_2 is small:

$$P_1 + \quad P_2 / D \quad p'^2 \quad c_2 [2p'q_1 + p''q_1^2].$$

$P_1 + P_2$ is positive (D is negative, it is easy to calculate it). At some time when c_2 increases, c being fixed, P_1+P_2 is increasing. It increases towards the joint profit of the merged firm. The merger becomes profitable.

*Proof 2.*When c_2 is very close to the value of the price when the merger has occurred, the profit P_2 is of second order. If $\Delta c_2>0$, q_2 passes from $\Delta q_2>0$ to 0, the rationed demand decreases (quantity – Δq_3). One proves: $\Delta q_1-\Delta q_2+\Delta q_3<0$, therefore $\Delta q_3<\Delta q_2$. The profit of the firm of cost c increases. Then it does not vary, because it is the profit of the monopoly on the rationed demand. Finally the variation of the joint profit is positive.

5. CONSOLIDATION AND COURNOT COMPETITION (4): CONCLUSION.

We have supposed that the state of the sector is Nash equilibrium before and after the merger. There are two other states which are possible, low price and tacit collusion:

- Low price is when the demand has decreased after an economic change. The firms should return to equilibrium by steps (groping) since the equilibrium is stable. But they can also try a merger.
- Tacit collusion is when signs (not negotiation, which is forbidden) warn each firm that the other firms invest prudently. The firm which is warned invests prudently.

Therefore the price is high and the profits increase.

There are 9 occurrences. The profitability of merger can be shown in a matrix 3x3, where the box (i, j) represents the state i before the merger and the state j after the merger: 1 for Low Price, 2 for Nash Equilibrium, 3 for Tacit Collusion. It is robust to write Not Profitable in the box (2, 2) (it has been demonstrated). One fulfills the other boxes thanks to two directions, easier (the merger is easier) and more uneasy (the merger is more uneasy):

- Easier:↑and →
- More uneasy: ↓ and ←.

The matrix is shown in the Table I.

Table 1: PROFITABILITY OF MERGERS (LP: LOW PRICE, NE: NASH EQUILIBRIUM, TC: TACIT COLLUSION)

	LP	NE	TC
LP	Unprofitable	Perhaps profitable	Probably profitable
NE	Unprofitable	Unprofitable	Perhaps profitable
TC	Unprofitable	Unprofitable	Unprofitable

From this matrix we deduce another matrix the "matrix of the regulator" which shows the behavior of the regulator in each case (Table 2).

Table 2: THE MATRIX OF THE REGULATOR

	LP	NE	TC
LP	Watching	Soft measures	Hard measures
NE	Watching	Watching	Soft measures
TC	Watching	Watching	Watching

Here “soft measures” means conditions such as to host a virtual operator, to sell assets (spectrum) etc. “Hard measures” means to allow an entry if the price rises. Consider the box (1, 3): a merger could be profitable because at the start there is the state of Low Price (therefore the profits are low) and after the merger the state of Tacit Collusion (therefore the profits are high). In other words the merged firm banks on high price after the merger to make the merger profitable. But the threat of an entry if the regulator decides it because the price rises could discourage the merger. When firms avoid merger they can benefit from no entry [6].

6. THE TOPIC OF THE STABILITY OF THE EQUILIBRIUMS.

One has to define a stable equilibrium.

For any succession of steps like E_1, E_2, E_1, E_3 etc. representing the competitors E_i optimizing their profit alternately, and any point M (p_1, p_2, p_3) at the start, there is convergence towards Nash equilibrium (p_{10}, p_{20}, p_{30}). Small variations are considered.

Concerning the Cournot equilibrium one demonstrates that it is stable when the demand is linear. One considers the successive “steps” and one uses the triangle inequality.

Concerning the Bertrand equilibrium there is a necessary and sufficient condition for the equilibrium with three competitors being stable: Det M < 0, where M is a matrix 3 x 3: M = (a_{ij}), with $a_{ij} = \partial P_i / \partial p_i \partial p_j$ (p_{10}, p_{20}, p_{30}).

An example of stability of the Nash equilibrium with Bertrand competition is when the demands are linear. The determinant is:

$$\begin{vmatrix} -2a & a1 & a2 \\ b1 & -2b & b2 \\ c1 & c2 & -2c \end{vmatrix}$$

All the coefficients a, a_1, a_2, b_1, b …are positive and –a, a_1, a_2, b_1, -b …are the $\partial D_i / \partial p_j$. One calls the diagonal terms OD (on the diagonal) and the terms not on the diagonal NOD (not on the diagonal). One demonstrates that on each line the terms NOD are less than the term OD, in absolute value. To prove it one considers a small move when p_3 is fixed: d p_1 = d p_2 = d p > 0. It is obvious that for E_1 customers lack after the move. One considers this move in axes O u_1 u_2 u_3 where the utilities of the consumers are represented (u_i being the utility of the product of the firm E_i). Therefore: d D_1 < 0, and $-a + a_1 < 0$. To calculate the sign of the determinant is straightforward.

Concerning a demand which is not linear, one easily demonstrates that the same inequality holds. Therefore the determinant is negative and the equilibrium is stable. One has to make the hypothesis $\partial^2 D_i / \partial p_i \partial p_j < 0$, which is justified in the chapter 6 of the quoted book.

One has also demonstrated that when the demand is linear, a unique solution exists: the determinant being negative the margins p_i-c_i are positive. Also, the equilibrium is “interior” (the demands are positive)[2].

To sum up:

- when the demands are linear the Nash equilibrium (with three competitors) exists and is stable.
- In the general case, if the Nash equilibrium exists, it is stable.

It is better to consider stable equilibriums when n = 3 (or n > 3). The Nash equilibrium has a concrete meaning: the competitors reach the equilibrium after some groping (several steps) no matter the order of these steps (it can be E_1, then E_2, then E_3 etc. or E_3, then E_2, then E_1 etc.).

7. CONSOLIDATION AND BERTRAND COMPETITION (1): THE TOPIC OF PROFITABILITY OF MERGERS.

The topics of Cournot competition and Bertrand competition are very different. In case of Cournot competition the firms choose their quantity [7]. In case of Bertrand competition the firms choose the price [8].

The topic of the profitability of a merger in case of Bertrand competition and n = 3, is awkward. Indeed, one is obliged to restrict the study to a particular case: when the merging firms are symmetrical. That is to say, the costs are equal ($c_1 = c_2$) and the demands are symmetrical (D_1 (p_1, p_2, p_3) = D_2 (p_2, p_1, p_3)). In this case, one demonstrates that: if a profitable merger is possible which corresponds to a unique Nash equilibrium (1) the three prices increase and (2) the profit of the third firm increases.

This kind of merger is profitable to the firms since the two profits increase (the joint profit and the profit of the third firm). But when the three prices increase, it is bad from the point of view of the regulator, since it is detrimental to consumers.

One starts considering the plane $p_3 = 0$ where are the point O (supposed to be the Nash equilibrium when there are three competitors) and the point M where the joint profit is maximized.

One imagines a small move d p_3, d p_3 > 0. The coordinates of the point M become p_1 + d p_1, p_2 + d p_2. The d p_1 and d p_2 are given in function of d p_3 by the two equations:

$$\partial^2 P_1+P_2 / \partial p_1^2 \, d p_1 + \partial^2 P_1+P_2 / \partial p_1 \partial p_2 \, d p_2 = - \partial^2 P_1+P_2 / \partial p_1 \partial p_3 \, d p_3$$

$$\partial^2 P_1+P_2 / \partial p_1 \partial p_2 \, d p_1 + \partial^2 P_1+P_2 / \partial p_2^2 \, d p_2 = - \partial^2 P_1+P_2 / \partial p_2 \partial p_3 \, d p_3$$

The determinant D is positive: it is a hessian determinant.

The formula for d p_1 and d p_2 are: (3)

$$D \, dp_1 = [- \partial^2 P_1+P_2 / \partial p_1 \partial p_3 \;\; \partial^2 P_1+P_2 / \partial p_1^2 + \partial^2 P_1+P_2 / \partial p_2 \partial p_3 \;\; \partial^2 P_1+P_2 / \partial p_1 \partial p_2] \, d p_3$$

$$D \, d p_2 = [\partial^2 P_1+P_2 / \partial p_1 \partial p_3 \, \partial^2 P_1+P_2 / \partial p_1 \partial p_2 - \partial^2 P_1+P_2 / \partial p_2 \partial p_3 \, \partial^2 P_1+P_2 / \partial p_1^2] \, d p_3$$

One knows the signs of some coefficients: the coefficients on the diagonal are negative (the function $P_1 + P_2$ is concave, to have a single maximum); the two other are positive (because of the strategic complements). But the signs of $\partial^2 P_1+P_2 / \partial p_1 \partial p_3$ and $\partial^2 P_1+P_2 / \partial p_2 \partial p_3$ are not known. So the signs of d p_1 and d p_2 are not known.

Now one considers p_3 which corresponds to the new value of p_3 (p_1, p_2): it is the value of p_3 maximizing P_3 when a point is in M ($\partial P_3 / \partial p_3 = 0$, p_1 and p_2 being he coordinates of M).

[2] The condition is D_i (c_1, c_2,c_3) > 0.

One writes:
$\partial^2 P_3 / \partial p_3 \partial p_1 \, d p_1 + \partial^2 P_3 / \partial p_3 \partial p_2 \, d p_2 + \partial^2 P_3 / \partial p_3^2 \, p_3 = 0$.

If one replaces p_3 by $d p_3$ one obtains $/ D \, d p_3$, being the determinant of the matrix:
$A_{1j} = \partial^2 P_1 + P_2 / \partial p_1 \partial p_j$
$A_{2j} = \partial^2 P_1 + P_2 / \partial p_2 \partial p_j$
$A_{3j} = \partial^2 P_3 / \partial p_3 \partial p_j$
One can write:
$/ D \, d p_3 = \partial^2 P_3 / \partial p_3^2 \, [d p_3 - p_3]$ (4)

Now one makes the hypothesis of two symmetrical firms, E_1 and E_2 (the merging firms): $c_1 = c_2$, and the demand are symmetrical ($D_1 (p_1, p_2, p_3) = D_2 (p_2, p_1, p_3)$).
If the demand is linear, the determinant is constant and not equal to 0. A necessary and sufficient condition for a profitable merger is < 0.
To demonstrate that, one considers the two contour lines:

- One passes by O and corresponds to the value of p_3 such that M is in the plane p_3 (here $p_3 = 0$).
- The other passes by M and corresponds to the value p'_3 which is the third coordinate of the point of the surface $\partial P_3 / \partial p_3 = 0$, at the vertical of M.

The contour lines have a negative slope, and the values are increasing towards the North East (it is a consequence of the strategic complements).
Therefore the contour lines can "catch up" when p_3 increases because < 0 implies $\Delta p_3 < dp_3$. If $d p_3 < 0$, $\Delta p_3 > dp_3$ and the contour lines cannot "catch up" one with the other. There is no possibility of equilibrium.
This equilibrium corresponds to an increase of $P_1 + P_2$ (the merger is profitable)
At the opposite, if > 0, any equilibrium will involve $P_1 + P_2$ decreasing (the merger is not profitable). If > 0 and $d p_3 > 0$, there is $p_3 > d p_3$ and the contour lines cannot "catch up" one with the other. If $d p_3 < 0$, $\Delta p_3 < dp_3$ and the contour lines could "catch up" and an equilibrium could exist. But it cannot be a profitable merger. Before the "catching up" the point M will meet the contour line passing by O. So the point M which is always on the first bisector (because of symmetry) will pass by O. The joint profit has the same value that when there is Nash equilibrium with three competitors. And p_3 is less than 0. More, the price p_3 will continue to decrease, before the Nash equilibrium is reached.
When p_3 decreases, it triggers the decrease of the maximal joint profit in the plane of third coordinate p_3. When the merger is profitable the three prices rise.
Also, the profit P_3 of the third firm increases.
To demonstrate it one can imagine a point starting from O and following the first bisector as far as the point at the vertical of the equilibrium point. Another point is at the vertical of this point and on the surface $\partial P_3 / \partial p_3 = 0$. At the start the profit P_3 is equal to its value when there is the Nash equilibrium with three competitors, then it increases since the move $d p_3$ has not effect, and the two prices p_1 and p_2 increase (in the plane $p_3 = k$ the point is at the South West of the point corresponding to the maximum of $P_1 + P_2$, before meeting this point). When the point arrives at the point of equilibrium (the competitors being the merged firm and the third firm), the profit P_3 has increased.
When the demand is not linear, < 0, in any point or at least along the curve ($p_1 (p_3)$, $p_2 (p_3)$, p_3), the prices p_1 and p_2 maximizing the joint profit when p_3 is fixed, is a sufficient condition. The merger is profitable, the three prices rise and the profit of the third firm increases.
In the general case, one has demonstrated that there are two kinds of mergers:

- Those which are profitable. The three prices rise. The profit of the third firm increases.
- Those which are not profitable. The three prices decrease. The profit of the third firm decreases.

If the demands are linear the determinant is negative or positive:

- If $\Delta<0$, there is a unique equilibrium (with the merged firm and the third firm) , the merger is profitable, the three prices increase and the profit of the third firm increases.
- If $\Delta>0$, there is no equilibrium.

If the two firms are symmetrical and the demands are linear, one demonstrates that $\Delta<0$. And if the demands are not linear one does not know the sign of because the sign of $\delta_{13}=\delta_{23}$ is not known. If this coefficient is negative it is an interesting particular case: $\Delta<0$ and the equilibrium exists and the merger is profitable. The formula (4) shows that the contour lines meet because they go one towards the other: that in O goes towards the North East and that in M goes towards the South West.
Concerning the existence of equilibrium, if $\Delta<0$ and if the demands are deduced from the utilities, it is sure that it exists. The utilities are finite. For instance they are in a cube $0 \le p_1 \le 1$, $0 \le p_2 \le 1$, $0 \le p_3 \le 1$. Therefore each profit is majored. When there are successive steps, $P_1 + P_2$ has successive, increasing values. And it is majored. There is a limit. By continuity it is the equilibrium.

8. CONSOLIDATION AND BERTRAND EQUILIBRIUM (5): AN EXAMPLE OF "BUY AND CLOSE DOWN" WHICH IS PROFITABLE WHEN n = 3.

In this example the utilities of the consumers are distributed in this way:

- They are in a plane $p_3 = 1 / 2$
- In this plane O' p_1 p_2 they are on the first bisector between O' and the point (1, 1) with a homogeneous density $1 / \sqrt{2}$

The costs are equal to 0.
The density is represented in the Figure 2.

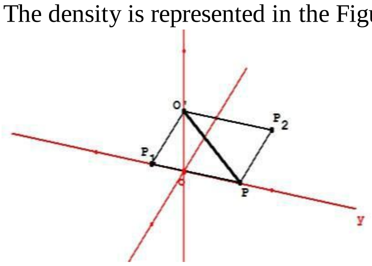

Figure 2: The density of the utilities in the example quoted in the paragraph 8 is shown. The density is on the segment O' P, which is in the plane $p_3 = 1 / 2$.

The Nash equilibrium with three competitors is described:
$p_1 = p_2 = 0$
$p_3 = 1 / 4$
$P_1 = P_2 = 0$
$P_3 = 1 / 16$

In case of "buy and close down" if E_1 has purchased E_2 the Nash equilibrium is described:
$p_1 = p_3 = 1 / 2$
$P_1 = 1 / 4$
$P_3 = 1 / 4$.
After the "buy and close down" the joint profit is 1 / 4 which is more than the joint profit when there are three competitors (0). Therefore the "buy and close down" is profitable.
The explanation is that the "buy and close down" allows breaking the Bertrand paradox. The Bertrand paradox is a particular case of Bertrand competition. It is when the utilities are on the first bisector (in case n = 2). If the two costs are equal (for instance they are equal to 0) a reasoning shows that Nash equilibrium corresponds to $p_1 = c$ and $p_2 = c$ ($p_1 = 0$ and $p_2 = 0$ if $c = 0$). The two firms make no profit [9]. Of course if one firm purchases the other the Bertrand paradox is broken and the purchase is profitable. This example is simple. It is passing from two competitors to monopoly. In the example which has been presented, there are three competitors, then two after the "buy and close down". Operations of the kind "buy and close down" occur.
For instance:

- Around 1900, the British company Red Sea Mining Company owned the single site where peridots (a kind of precious stone) were produced, in the island of Zabargad. The company bought an Italian island where peridots were produced then closed down the company producing them. The story is told by a French adventurer, Henri de Monfreid, in his memoirs ("Les secrets de la Mer Rouge", "The secrets of the Red Sea"). He delivered a rough geography of the region.
- In France the carmaker Panhard was bought by Citroen in 1965 and closed down in 1967. It ceased to produce cars.
- In France, in 2016 a company of bus transport, Megabus, was bought by a competitor, Flixbus, then closed down.

9. CONCLUSION

This conclusion is in three parts, resuming the results, reflection on the method and proposals for the following:
Resuming the results.
The regulator is more secure when modeling thanks to Cournot competition can be used. If there are three large, stable operators he does not fear attempts of consolidation. Concerning side payments, they can make a merger profitable, but there are complex negotiations. The merger is acceptable in case of "strategic reasons" (for instance, a project involving synergies). When the model of Bertrand competition can be used, it is more awkward. A profitable merger can make the prices increase. And a clear criterion for the relevant market lacks. Portfolios of brands in fields like shoes, clothes, fashion … have appeared. In the past, in the USA, the Clayton Act prohibited an owner to own more than one firm selling goods on a market [10].
Reflection on the method.
The interesting points are: (1) game theory obliges to pay attention to the direct effect and the strategic effect (2) some results are in accordance with intuition, other are counterintuitive like examples of joint profit increasing when one of the costs increases, or examples of total profit decreasing when a merger occurs (3) one supposes players rational and shortsighted. They adjust their choice, maximizing their profit, when they know the choices made by the other players. In case of Cournot competition, they know the structure of the price, the price and their quantity, so they deduce the quantity sold by the other and know their rationed demand. In case of Bertrand competition, they know the prices. Therefore it is interesting if the equilibriums are stable. But one has to suppose an ability to anticipate in case of merger. Otherwise one cannot write "in case of Bertrand competition, if the managers choose merger, the merger is profitable and the prices increase". One supposes that the managers anticipate that the merger is profitable.
Proposals for the following.
The topic of buy "and close down" should be more studied. There are interesting details. Sometimes assets are kept and the other are closed down. Or the useless assets are resold. If assets are resold it can make the "buy and close down" more profitable.

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