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An Efficient Hybrid Conjugate Gradient Coefficient under Inexact Line Search

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ABSTRACT

The conjugate gradient (CG) methods are iterative procedures for unconstrained optimization problem. Many variants and alterations have been done lately to develop this method. In this research, we suggest a new hybrid CG coefficient β_k by combining a modified Hestenes-Steifel (HS) formula with Fletcher and Reeves (FR). Theoretic proves has shown that the new technique achieves sufficient descent condition if inexact line search (strong Wolfe-Powell) is used. Besides, most of the numerical outcomes show that our technique is very efficient when compared to HS, FR and some famous hybrid CG for given standard test problems. The numerical outcomes also displayed that the new coefficient β_k performs better than the original HS and FR methods.

Key words: Hybrid CG, unconstrained optimization, inexact line search, sufficient descent condition.

1. INTRODUCTION

The CG method (CGM) is among the efficient methods for solution of unconstrained optimization problems. Its convergence properties and simplicity make it one of the best methods in real life application like economics, health and physics. The CG method is designed to solve problem in the form:

$$\min f(x), \quad x \in R^n \tag{1}$$

where $f: R^n \to R$ is smooth and its first derivative is defined by $g(x) = \nabla f(x)$. The nonlinear CGM obtain a sequence by using the following recurrence formula;

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0,1,2,\ldots \tag{2}$$

where α_k is the step size computed by a method called line search, x_k is the current iterate and d_k is the direction of search determined by.

$$d_k = \begin{cases} -g_k & , \text{ if } k = 0; \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1 \end{cases}, \tag{3}$$

The step size α_k in CGM is computed to satisfy some form of Wolfe conditions [1]. The standard Wolfe condition is:

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k \tag{4}$$

$$g_{k+1}^T d_k \geq \sigma g_k^T d_k \tag{5}$$

where $0 < \delta \leq \sigma < 1$. While order CG method are computed to satisfy the strong Wolfe Powell (SWP) is computed by (4) and

$$|g_{k+1}^T d_k| \leq -\sigma g_k^T d_k \tag{6}$$

In the linear CGM or nonlinear CGMs the parameter β_k is called CG coefficient [2], Some of the famous CG formula for β_k are:

$$\beta_k^{HS} = \frac{g_k^T(g_k - g_{k-1})}{(g_k - g_{k-1})^T d_{k-1}}$$ (Hestenses and Stiefel [3])

$$\beta_k^{FR} = \frac{g_k^T g_k}{g_{k-1}^T g_{k-1}}$$ (Fletcher and Reeves [4])

$$\beta_k^{PRP} = \frac{g_k^T(g_k - g_{k-1})}{g_{k-1}^T g_{k-1}}$$ (Polak, Ribiere and Polyak [5, 6])

$$\beta_k^{CD} = \frac{g_k^T g_k}{d_{k-1}^T g_{k-1}}$$ (Conjugate Descent [7])

$$\beta_k^{LS} = -\frac{g_k^T(g_k - g_{k-1})}{d_{k-1}^T g_{k-1}}$$ (Liu and storey [8],)

$$\beta_k^{DY} = \frac{g_k^T g_k}{(g_k - g_{k-1})^T d_{k-1}}$$ (Dai and Yuan, [9])

The convergence analysis of these algorithms with different α_k was studied by several academics. (see Zoutendijk [10], Powell [11,12], Z. Wei [13], Zhi- Feng Dai [14], Al-Baali [15], Dai and Yuan [16], Narene et al., [29]).

However, the convergence analysis is yet to be proved for PRP, LS and HS methods under all stated line searches [30, 31]. The foremost purpose is that they cannot guarantee the decency of values of objective function at every iteration [17].

Some well-known CGMs methods have strong convergence property like FR, DY, and CD, but they may not perform well. Others like PRP, HS, and LS perform well but they may not converge. So, the hybrid CGMs are formulated to combine the attractive features of the well-known CGMs.

Recently, Touati and Storey [18] improve AL-Baali's [15] formula for the FR method

$$\beta_k \in \{0, \beta_k^{FR}\}$$

and propose the first hybrid CG algorithm

$$\beta_k = \begin{cases} \beta_k^{PRP} & if \quad 0 \le \beta_k^{PRP} \le \beta_k^{FR} \\ \beta_k^{FR} & , \quad else\ where \end{cases}$$

Motivations by this, Gilbert and Nocedal [19] extended Al-Baali's formula for β_k

$$\beta_k \in \{-\beta_k^{FR}, \beta_k^{FR}\}$$

to propose the formula

$$\beta_k^{HGN} = max\{-\beta_k^{FR}, min\{\beta_k^{PRP}, \beta_k^{FR}\}\}.$$

Hu and Storey [20] further suggested an improved CG formula as follows:

$$\beta_k^{HHUS} = max\{0, min\{\beta_k^{PRP}, \beta_k^{FR}\}\}.$$

Dai and Yuan [21], propose a family of CGM that are globally convergent and defined by

$$\beta_k = \frac{\|g_k\|^2}{\lambda\|g_{k-1}\|^2 + (1-\lambda) d_{k-1}^T y_{k-1}}$$

where $\lambda \in [0,1]$ is parameter and $y_{k-1} = g_k - g_{k-1}$.

Recently, Xiao and Kong [22], combine β_k^{DY} and β_k^{HS}, as follows.

$$\beta_k^1 = \begin{cases} a_1\beta_k^{DY} + a_2\beta_k^{HS} & if \quad \|g_k\|^2 > |g_k^T g_{k-1}| \\ 0 & , \quad else\ where \end{cases}$$

More recently, a new hybrid CG is considered by Djordjevic [23]. The CG parameter β_k which is a combination of β_k^{CD} and β_k^{LS} :

$$\beta_k^{hyb} = (1-\theta_k).\beta_k^{LS} + \theta_k\beta_k^{CD}$$

where the conjugacy condition is satisfied if the parameter θ_κ is computed in such away.

2. A NEW FORMULA FOR β_κ

Numerous researchers have tried to improve the conjugate gradient methods. This has led to the development of many variants of CG algorithms. For instance, a modification of the PRP algorithm known as WYL method was proposed by Wei et al. [24]. Recently Zhang [17] extended the work of WYL to propose a variant of PRP known as NPRP that fulfilled descent condition and satisfied the global convergence property under inexact line search. More so, Dai and Wen [14] suggested the DPRP method which is an efficient modification of the NPRP method.

Motivated by nice convergence properties and efficient numerical results of these methods [18, 24], we suggest a new formula for β_k called β_k^{HTM}, where HTM indicates Hybrid Tala't and Mustafa.

$$\beta_k^{HTM} = \begin{cases} \beta_k^{TMR}, \ if\ \|g_k\|^2 > \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}| \\ \beta_k^{FR}, \quad other\ wise \end{cases} \quad (7)$$

where the new β_k^{TMR} is modification of HS method as follow:

$$\beta_k^{TMR} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|}{d_{k-1}^T(g_k - g_{k-1})}$$

The algorithm of the proposed method is defined as follows.

Algorithm (Algorithm for HTM)

Stage 1: Initialization. *let* $k = 0$ and $x_0 \in R^n$, set $d_0 = -g_0$. If $g_0 = 0$ then stop.
Stage 2: Calculate the stepsize α_k by (SWP) line search.
Stage 3: Set $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = g(x_{k+1})$ if $\|g_{k+1}\| \le \varepsilon$ then stop.
Stage 4: Calculate β_k by (7), and produce d_{k+1} by (2).
Stage 5: Let $k = k+1$ go to Stage 2.

Lately Wei et al. [26] introduce a variation of PRP coefficient referred to the WYL method.

$$\beta_k^{WYL} = \frac{g_k^T g_k - g_k^T g_{k-1}\frac{\|g_k\|}{\|g_{k-1}\|}}{\|g_{k-1}\|^2}$$

Enlightened by preceding ideas [3], [26], we introduce our β_k which is identified as $\beta_k^{TM^*}$,where TM* symbolizes Tala't and Mustafa. The new $\beta_k^{TM^*}$ is a variation of HS method which is as follows:

$$\beta_k^{TM^*} = \frac{g_k^T(m(g_k - g_{k-1}))}{m(g_k - g_{k-1})^T d_{k-1}}, where\ m = \frac{\|g_{k-1}\|}{\|g_k\|} \quad (5)$$

The algorithm of the proposed coefficient is as follows:

Algorithm 1

Stage 1: Initialization. Given $x_0 \in R^n, \varepsilon \ge 0,$ set $d_0 = -g_0$ if $\|g_0\| \le \varepsilon$ then stop.
Stage 2: Compute α_k by (3).
Stage 3: Let $x_{k+1} = x_k + \alpha_k d_k$, $g_{k+1} = g(x_{k+1})$ if $\|g_{k+1}\| \le \varepsilon$ then stop.
Stage 4: Calculate β_k by (5), and produce d_{k+1} by (4).
Stage 5: let $k = k+1$ go to Stage 2.
either SI (MKS) or CGS as primary units. (SI units are strongly encouraged.) English units may be used as secondary

3. CONVERGENCE ANALYSIS

The convergence analysis is among important condition to consider when developing a new conjugate gradient coefficient. In addition to numerical performance, the proposed method should be able to satisfy the sufficient descent condition defined by

$$g_k^T d_k \leq -c\|g_k\|^2 , \quad \forall k \geq 0 , \qquad (8)$$

where $c \in (0,1)$.

In this section, we need to show that the proposed method satisfies the sufficient descent condition. The following Theorem is very important in the proof.

Theorem

Suppose that the sequences $\{g_k\}$ and $\{d_k\}$ are derived by a method of the form (2), (3) and (7), and α_k is computed by the SWP line search defined by (4) and (6), if $g_k \neq 0$, then (8) holds.

Proof

The HTM method in (7) can be considered in two cases:

Case I: If $\|g_k\|^2 < \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|$, $\beta_k^{TMR} < 0$, then β_k^{HTM} automatically returns β_k^{FR}. Then the sufficient descent condition given in (8) holds [4].

Case II: If $\|g_k\|^2 < \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|$, then $\beta_k^{HTM} = \beta_k^{TMR}$. To prove that the sufficient descent condition given in (8) holds, we need to simplify β_k^{TMR} to ease the theoretical proof. From (6), we have

$$\beta_k^{TMR} = \frac{\|g_k\|^2 - \frac{\|g_k\|}{\|g_{k-1}\|}|g_k^T g_{k-1}|}{d_{k-1}^T (g_k - g_{k-1})}$$

$$\leq \frac{\|g_k\|^2}{{d_{k-1}}^T (g_k - g_{k-1})} \leq \frac{\|g_k\|^2}{\|g_{k-1}\|^2}$$

Hence, we obtain

$$\beta_k^{TMR} \leq \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \qquad (9)$$

Using (6) and (9), we get

$$|\beta_{k+1}^{TMR1} g_{k+1}^T d_k| \leq \frac{\|g_{k+1}\|^2}{\|g_k\|^2}\sigma|g_k^T d_k| \qquad (10)$$

From (3)

$$\frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} = -1 + \beta_{k+1}\frac{g_{k+1}^T d_k}{\|g_{k+1}\|^2} \qquad (11)$$

By induction, we can prove the descent property of $\{d_k\}$. Since $g_0^T d_0 = -\|g_0\|^2 < 0 , if\ g_0 \neq 0$, now suppose that $d_i , i = 1,2, \dots , k$ are descent directions, i.e $g_i^T d_i < 0$

From (10), we have

$$\frac{\|g_{k+1}\|^2}{\|g_k\|^2}\sigma g_k^T d_k \leq \beta_{k+1}^{TMR1} g_{k+1}^T d_k \leq -\frac{\|g_{k+1}\|^2}{\|g_k\|^2}\sigma g_k^T d_k$$

$$\frac{\sigma g_k^T d_k}{\|g_k\|^2} \leq \frac{\beta_{k+1}^{TMR1} g_{k+1}^T d_k}{\|g_{k+1}\|^2} \leq -\frac{\sigma g_k^T d_k}{\|g_k\|^2}$$

From (11),

$$\frac{\sigma g_k^T d_k}{\|g_k\|^2} \leq 1 + \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -\frac{\sigma g_k^T d_k}{\|g_k\|^2}$$

By repeating this procedure and the fact $g_0^T d_0 = -\|g_0\|^2$, we get

$$-\sum_{j=0}^{k}\sigma^j \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -2 + \sum_{j=0}^{k}\sigma^j \qquad (12)$$

Since,

$$\sum_{j=0}^{k}\sigma^j < \sum_{j=0}^{\infty}\sigma^j = \frac{1}{1-\sigma}$$

then, (12) is rewritten as

$$-\frac{1}{1-\sigma} \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -2 + \frac{1}{1-\sigma} \qquad (13)$$

Restricting $\sigma \in (0,0.5)$, we obtain $\frac{1}{1-\sigma} < 2$. Let $c = 2 - \frac{1}{1-\sigma}$, then $0 < c < 1$, and (13) become

$$c - 2 \leq \frac{g_{k+1}^T d_{k+1}}{\|g_{k+1}\|^2} \leq -c$$

$$g_k^T d_k \leq -c\|g_k\|^2$$

This implies that (8) holds. The proof is complete. □

4. NUMERICAL RESULTS

To ascertain the effectiveness of the proposed HTM, we compare HTM method with the methods of FR, HS, HHUS and HGN using 33 randomly selected test functions considered from Andrei [25]. The initial points are chosen randomly beginning from a point near to the solution point to points far away with dimensions $2 \geq 1000$. In certain cases, the computations fail to obtain the step size and thus, the method is unsuccessful [27, 28]. Numerical outcomes are compared based on CPU time and number of iteration (NOI) and the performance analyzed using performance profile by Dolan and More [26] as in Figure 1 and Figure 2. All computations are carried out on an Intel (R) Core TM i3-M350 (2.27GHz) CPU processor, with 4 GB RAM memory.

Table 1: List of test problems

NO	Function	Dim	Initial point
1	SIX HUMP	2	(0.5,0.5), (8,8), (40,40)
2	THREE HUMP	2	(-1,1),(-2,2),(2,-2)
3	LEON	2	(2,2),(4,4),(8,8)
4	QUADRATIC QF1	2	(3,3),(5,5),(10,10)
5	MATYAS	2	(5,5),(10,10),(15,15)
6	DIAGONAL 2	2	(1,1),(5,5),(15,15)
7	BOOTH	2	(10,10),(25,25),(100,100)
8	RAYDAN	2	(3,3),(13,13),(22,22)
9	ZETTL	2	(5,5),(20,20),(50,50)
10	TRECANNI	2	(5,5),(10,10),(50,50)

11	NONDIA	2	(10,10),(20,20),(35,35)
12	HAGER	2	(7,7),(15,15),(20,20)
13	EXTENDED MARATOS	2	(10,10),(60,60),(120,120)
14	EXTENDED PENALTY	2	(40,40),(80,80),(100,100)
15	GENERLIZED TRIDIAG 1	2	(3,3),(21,21),(90,90)
16	QUADRATIC QF2	2	(4,4),(40,40),(80,80)
17	CLOVILLE	4	(2,..,2),(4,..,4),(10,..,10)
18	EXTENDED WOOD	4	(5,..,5),(20,..,20),(30,..,30)
19	DIXON & PRICE	2 4	(6,6),(18,18),(60,60) (6,.,6),(18,..,18),(60,..,60)
20	ARWHEAD	2 10	(8,8),(24,24),(32,32) (8,.,8),(24,..,24),(32,..,32)
21	GENERRALIZE D QUARTIC	2 10	(7,7),(70,70),(140,140) (7,..,7),(70,..,70),(140,..,140)
22	FLETCHCR	2 10,100,1000	(12,12),(15,15),(35,35) (12,..,12),(15,..,15),(35,..,35)
23	ROSENBROCK	2 10,100,1000	(3,3),(15,15),(75,75) (3,..,3),(15,..,15),(75,..,75)
24	SHALLOW	2 10,100,1000	(2,2),(12,12),(200,200) (2,..,2),(12,..,12),(200,..,200)
25	EXTENDED WHITE & HOLST	2 10,100,1000	(3,3),(6,6),(10,10) (3,..,3),(6,..,6),(10,..,10)
26	EXTENDED BEALE	2 10,100,1000	(-4,-4),(-1,-1),(4,4) (-4,..,-4),(-1,..,-1),(4,..,4)
27	PERTURBED QUADRATIC	2 10,100,1000	(1,1),(5,5),(10,10) (1,..,1),(5,..,5),(10,..,10)
28	EXTENDED TRIDIAGONAL 1	2 10,100,1000	(25,25),(50,50),(75,75) (25,..,25),(50,..,50),(75,..,75)
29	DIAGONAL 4	2 10,100,1000	(1,1),(20,20),(40,40) (1,.,1),(20,..,20),(40,..,40)
30	EXTENDED HIMMELBLAU	2 10,100,1000	(10,10),(50,50),(125,125) (10,..,10),(50,..,50),(125,.,125)
31	EXTENDED DENSCHNB	2 10,100,1000	(5,5),(30,30),(50,50) (5..,5),(30,..,30),(50,..,50)
32	EXTENDED BLOCK DIAG BD1	2 10,100,1000	(1,1),(5,5),(10,10) (1,..,4),(5,..,5),(10,..,10)
33	SUM SQUARES	2 10,100,1000	(1,1),(5,5),(10,10) (1,..,1),(5,..,5),(10,..,10)

In [26] Dolan and More suggested an ideal to assess and equate the performance of the set of solvers S on a test set of problems P. Assuming n_s solvers and n_p problems exists, then for each problem p and solver s, they defined

$t_{p,s}$ = computing time (NOI. or CPU time) necessary to solve problems p by solver s. Wanting a fixed form for evaluations, they equated the performance of problem p by solver s with the finest performance for any solver to the same problem using the ratio

$$r_{p,s} = \frac{t_{p,s}}{min\{t_{p,s} : s \in S\}}$$

Assume that a parameter $r_M \geq r_{p,s} \quad \forall\, p, s$ is selected, and $r_M = r_{p,s}$ iff solver s does not solve problem p. The performance of solver s on any given problem could be of concern, but since we want to achieve a general valuation of the performance of the solver, it was defined

$$p_s(t) = \frac{1}{n_p}\{p \in P : r_{p,s} \leq t\}$$

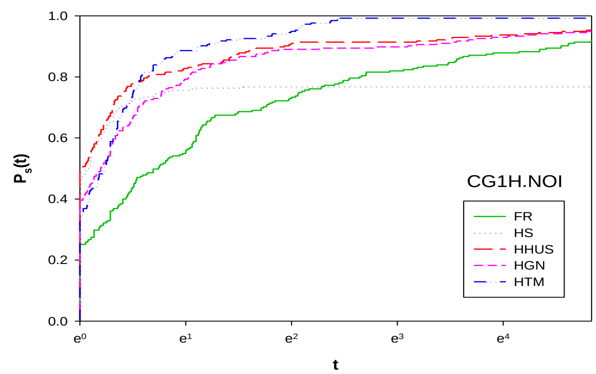

Figure 1: Performance profile based on NOI

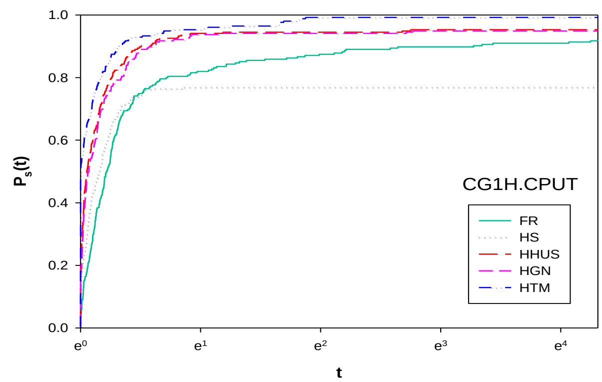

Figure 2: Performance profile based on CPU time

In Figure 1 and Figure 2, the performances of HTM is compared to the known methods of HS and FR and hybrid methods of HHUS and HGN. The results show that new hybrid method has shown considerable improvements from the FR and HS methods by while retaining their attractive featuress. Besides, it can be said that HTM succeeds to outperforms HHUS and HGN. In summary, the figures indicate that HTM is robust and competitive to other CGMs.

5. CONCLUSION

In this paper, we studied a new hybrid method for unconstrained optimization problems. We showed that the suggested method achieves the sufficient descent condition under inexact (SWP) line search. The outcome of the numerical tests shows that the new method is efficient when compared with other CGMs. The authors recommend testing this method under other line searches in future research.

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