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General Relativistic Magnetohydrodynamic Source Terms in 3+1 Form

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ABSTRACT

The main purpose of this paper is to study the general relativistic magnetohydrodynamics source terms in 3+1 form. In this paper a set of equations, which are suitable for numerical interpretation in full 3+1 dimensions is determined. Section 2 is devoted to electromagnetic source terms in 3+1 form. In the section 3 we have delineated the general relativistic magnetohydrodynamics and obtained a condition that 3+1 source terms are in the same form as that in flat space. In the end we have established a theorem regarding general relativistic magnetohydrodynamics.

Key words : Gravito-electromagnetism, Magnetohydrodynamics, Relativistic, Source terms.

1. INTRODUCTION

Gravito-electromagnetism (GEM) is concerned with analogies between the electromagnetism and relativistic gravitation equations. It establishes a link between Maxwell's field equations under some approximation and the Einstein's field equations of general relativity [Mashhoon Bahram, 2008]. Gravitomagnetism is a commonly used term directly referring to the kinetic effects of gravity, in analogy with the magnetic effects of electric charge which is undergoing some motion. Gravitoelectromagnetic field strength does not result into a complete set of Maxwell-like equations [Ming Kian et al., 2017]. Magnetic fields are very important when considered for the determination of the evolution of various relativistic objects. Initially a weak magnetic field is applied which increases with time to significantly influence the dynamical behaviour of the object under consideration. Besides magnetic fields, if the gravitational field is also strong and dynamical, then magnetic fields acquire the capability of affecting the complete geometry of space-time. Consequently the terms which involve the magnetic and electric field becomes useful not only as electromagnetic forces acting on the matter in the equations of relativistic MHD but also as stress energy sources governing the metric in Maxwell's equations for gravitational field. T. S. Chauhan et al. have investigated electromagnetic field tensor in magnetohydrodynamical approximation [Chauhan T. S. et al, 2011]. It is worthwhile to mention here that Soret and Dufour effect in generalized MHD Couette flow behave like binary mixture [T. S. Chauhan et al., 2012] & [T. S. Chauhan et al., 2020].

In order to explain the notations and to clarify the concepts used, some basic formulae of the relativistic MHD are briefly presented here.

The extrinsic curvature in terms of time derivative of the spatial metric is considered as the second evolution equation.

$$\partial_t \gamma_{ab} = L_\beta \gamma_{ab} - 2\alpha k_{ab} \tag{1}$$

Wherein L denotes a symbol for the derivative with respect to g_{ab} and $k = \gamma^{ab} k_{ab}$ is the trace of the extrinsic curvature. The quantity ρ is the total mass energy density as measured by the normal observer and is given by

$$\rho = \eta_i \eta_j T^{ij} \tag{2}$$

Here S_α is the momentum density and is defined as

$$S_\alpha = -\gamma_{\alpha i} \eta_j T^{ij} \tag{3}$$

Where S_{ab} is the stress and is expressed as

$$S_{ab} = \gamma_{ai} \gamma_{bj} T^{ij} \tag{4}$$

Here S is defined as the trace of S_{ab} and is given by

$$S = S^{ab} \gamma_{ab} \tag{5}$$

Since the stress energy tensor T^{ij}_F for a perfect fluid can be expressed as

$$T_F^{ij} = \rho_0 h u^i u^j + P g^{ij} \tag{6}$$

Where P is the pressure, h is specific enthalpy and ρ_0 is the rest-mass density as measured by an observer moving with the fluid u^i. Specific enthalpy h is given by [Chauhan T. S. et al., Shodh Hastakshep, 2011]:

$$h = 1 + \epsilon + \frac{P}{\rho_0} \quad (7)$$

Wherein ϵ is the specific internal energy density.

The Faraday tensor F^{ij} is given by

$$F^{ij} = \eta^i E^j - \eta^j E^i + \epsilon^{ijk} B_k \quad (8)$$

Where B_i and E^j are magnetic field and electric field as observed by a normal observer η^i.

2. ELECTROMAGNETIC SOURCE TERMS IN 3+1 FORM

The fluid contribution to the source terms can be obtained by using the fluid stress energy tensor (6) into equation (2) to (5).

$$\rho_F = \rho_0 h W^2 - P \quad (9)$$

$$S_a^F = \rho_0 h W u_a \quad (10)$$

$$S_{ab}^F = P\gamma_{ab} + \frac{S_a^F S_b^F}{\rho_0 h W^2} \quad (11)$$

$$S_F = 3P + \rho_0 h (W^2 - 1) \quad (12)$$

The electromagnetic stress energy tensor T^{ij}_{EM} using Faraday tensor F^{ij} assumes the form

$$4\pi T_{EM}^{ij} = F^{ik} F_k^j - \frac{1}{4} g^{ij} F_{kl} F^{kl} \quad (13)$$

With the use of (8), we find

$$F_{ij} F^{ij} = 2(B_a B^a - E_a E^a) \quad (14)$$

Insertin $\epsilon^{ijk}\epsilon_i^{lm} = \gamma^{jl}\gamma^{km} - \gamma^{jm}\gamma^{kl}$, the first term of (13) takes the form

$$F^{ik} F_k^j = \eta^i \eta^j E_a E^a + 2\eta(i\epsilon j)klE_k B_l - E^i E^j - B^i B^j + \gamma^{ij} B_a B^a \quad (15)$$

The electromagnetic stress energy tensor in 3+1 form, after combining the (14) and (15) yields

$$4\pi T_{EM}^{ij} = \frac{1}{2}(\eta^i \eta^j + \gamma^{ij})(E_a E^a + B_a B^a) + 2\eta(i\epsilon j)klE_k B_l - (E^i E^j + B^i B^j) \quad (16)$$

The electromagnetic source terms can be obtained by inserting the stress energy tensor (6) into (2) to (5). The energy density of electromagnetic fields for the mass energy density ρ_{EM} is given by

$$4\pi \rho_{EM} = \eta_i \eta_j 4\pi T_{EM}^{ij} = \frac{1}{2}(E_a E^a + B_a B^a) = \frac{1}{2}(E^2 + B^2) \quad (17)$$

The Poynting vector obtained from the energy flux S_a^{EM} yields

$$4\pi S_a^{EM} = -\gamma_{ai}\eta_j 4\pi T_{EM}^{ij} = -\gamma_{ai}\eta_j \eta^j \epsilon^{ikl} E_k^l = \epsilon_{abc} E^b B^c = (\boldsymbol{E} \times \boldsymbol{B}) \quad (18)$$

The stress tensor S_{ab}^{EM} can be written as

$$4\pi S_{ab}^{EM} = \gamma_{ai}\gamma_{bj} 4\pi T_{EM}^{ij} = -E_a E_b - B_a B_b + \frac{1}{2}\gamma_{ab}(E^2 + B^2) \quad (19)$$

Finally the trace of (18) is equal to mass energy density

$$S_{EM} = \frac{(E^2 + B^2)}{8\pi} \quad (20)$$

Then in view of analogy with relations (17) and (20), we observe that

$$S_{EM} = \rho_{EM} \quad (21)$$

The above results are expressed in terms of the electromagnetic field components as observed by a normal observer η^i. It is worthwhile to mention here that by a normal observer we mean an observer which is at rest with respect to the slices Σ. From mathematical analysis we conclude that 3+1 source terms have the same form as in flat space. It is hoped that the results obtained will serve as a scientific tool for understanding more complex MHD problems and provide more useful information for engineering applications.

3. GENERAL RELATIVISTIC MAGNETOHYDRODYNAMICS IN 3+1 FORM

The equations of motion for the perfect fluid in absence of electromagnetic fields can be written from the local conservation of energy momentum [Anton Luis et al., 2008].

$$\nabla_j T^{*ij} = 0 \quad (22)$$

and conservations of baryons, is given by

$$\nabla_t(\rho_0 u^i) = 0 \quad (23)$$

Rest mass density variable is defined as

$$D = \rho_0 W \quad (24)$$

and internal energy density, variable is given by

$$E = \rho_0 e W \quad (25)$$

and a momentum variable is expressed as

$$S_i = \rho_0 h W u_i = (E + D + PW) u_i \quad (26)$$

The equation of continuity in terms of rest mass density variable is given by

$$\partial_b\left(\gamma^{1/2} D v^b\right) + \partial_t\left(\gamma^{1/2} D\right) = 0 \quad (27)$$

Transvecting with u^i, (24) assumes the form

$$\partial_t\left(\gamma^{1/2} E\right) + \partial_b\left(\gamma^{1/2} E V^b\right) = -P\left[\partial_t\left(\gamma^{1/2} W\right) + \partial_a\left(\gamma^{1/2} W V^a\right)\right] \quad (28)$$

This equation is known as energy equation and is given by

$$\partial_a\left(\gamma^{1/2} S_a\right) + \partial_b\left(\gamma^{1/2} S_a V^b\right) = -\alpha\gamma^{1/2}\left[\partial_a P \frac{S_i S_j}{2\alpha S^t}\partial_a g^{ij}\right] \quad (29)$$

Gamma-law equations of state, is expressed as

$$P = (\Gamma - 1)\rho_0 e \quad (30)$$

The right hand side of the energy (30) can be eliminated to yield

$$\partial_t\left(\gamma^{1/2} E_*\right) + \partial_b\left(\gamma^{1/2} E_* V^b\right) = 0 \quad (31)$$

Where E_* is the energy variable and is defined as

$$E_* = (\rho_0 e)^{1/\Gamma} . W \quad (32)$$

With the help of the relation $u_i u^i = -1$, we can obtain in the following way

$$W = \alpha u^t = (1 + \gamma^{ab} u_a u_b)^{1/2} \quad (33)$$

and V^a is given by

$$V^a = \frac{\alpha\gamma^{ab} u_b}{W} - \beta^a \quad (34)$$

Consequently follows

$$W = \frac{\alpha\gamma^{ab} u_b}{V^a + \beta^a} \quad (35)$$

In view of above facts, we have a theorem:

3.1 Theorem

For the general relativistic magnetohydrodynamics in 3+1 form the following relation

$(V^a + \beta^a) W u_a - \alpha(W^2 - 1) = 0$

holds good.

Proof:

Multiplying equation (34) by u_a on both sides, we obtain

$$\gamma^{ab} u_b u_a = \frac{W u_a (V^a + \beta^a)}{\alpha} \quad (36)$$

By virtue of (33) and (36), we get

$$W = \left[1 + \frac{W u_a(\beta^a + V^a)}{\alpha}\right]^{1/2} \quad (37)$$

Consequently yields

$$(V^a + \beta^a) W u_a - \alpha(W^2 - 1) = 0 \quad (38)$$

This establishes the theorem.

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