



Comparative Study of Collocation Method and Galerkin Method for Solving Nonlinear Partial Differential Equation

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ABSTRACT

A lot of essential consciences substantial and synthetic experience can be described by Partial Differential Equation (PDE). These Galerkin method (GM) and Collocation method (CM) are used to solve some examples of nonlinear Partial Differential Equation (PDE). The particular times is used in these methods because it can influence the collected result from the solution to be compared in terms of convergence study and the accuracy of the numerical solution. Error percentage is calculated of these both method to get the best method for solving nonlinear PDE. Maple 18 program have been advanced based on the analysis in order to solve the PDE.

Key words : Partial Difference Equation (PDE), Finite Element Method (FEM), Galerkin Method (GM), Collocation Method (CM).

1. INTRODUCTION

Single or a system of Partial Differential Equations (PDEs) is mostly encountered by us in many sciences and engineering fields. PDEs also describe many of the basic natural laws in physical or chemical phenomena [1]. The special case of the PDE is ordinary differential equation (ODE) but the behaviour of the solutions is different in general [2,13]. Partial derivative is when the function that depends upon several variables, that can differentiate with respect to either variable while holding the other constant. There are many examples of PDE equations, which are heat equation, wave equation, laplace equation and others [3]. ODE has linearly independent solutions of order m , but PDE has infinitely many arbitrary functions in the solutions [4]. Tawfik, M [5] stated that Galerkin Method (GM) and Collocation method (CM) is used in Finite Element Method (FEM). Suli, E [6, 15] explained about mathematical aspects of finite element approximation, including stability, accuracy, reliability and adaptivity. FEM as general tool for numerical solution partial differential equation. Interpolation techniques and iteration procedures are also evaluated to derive basic priori and aposteriori [6]. Cerdà, J. J [7] used FEM method known as Galerkin FEM.

This function to approximate solution to obey for each element approach in order to minimize the value of residual. The GM and CM is a generic class to approximate solution. These two methods are types of weighted residual methods [8]. The accuracy of second order FEM to solve the Burgers' equation. This Burgers equation is obtained on the whole real line and subjected to initial conditions with compact support. The finite element results solution with using Robin's boundary conditions approach the same solution that calculated by exact Dirichlet and Neumann boundary conditions. This allows to obtain realistic solutions in some cases. B.V. Siva Prasad Reddy, K. R [9] used the Galerkin FEM to solve the numerical solutions of Burgers equation problems with initial and boundary conditions. Chaudhari, T. U [10] stated that Galerkin FEM is to find the numerical solutions of advection-diffusion equation. Also, numerical and analytical solutions was compared and proposed scheme has good accuracy. Dinkar Sharma, R. J [11] discussed about the stability analysis of the collocation method in the Burgers equation. Free technique based on sinc functions was used in this problems. Zarebnia, M [12] studied the sensivity of the orthogonal collocation method to different parameters. The orthogonal collocation on finite elements is also used to PDE from chemical kinetics. Spectral-CM based on Legendre-Gauss-Lobatto point has presented a computational method for solving a class of fractional integral equation of the second kind. The numerical result are compared with the exact solutions .

1.1 Methodology

There are seven (7) step involved in this research

Step 1 : Define Project

The step discussed about the method used to find the accuracy of numerical results with exact solution for solving nonlinear PDE problems. The chosen method are GM and CM.

Step 2: Literature Study

This step is about the previous research papers, journals and books that related of this project. The important data regarding of these methods is collected. The solution of these methods has been determined.

Step 3: Identify formula and types of function

This step is explained about the identifying formula of selected methods. The studied about these two methods, GM and CM for more understanding. The chosen function applied to the methods to get the accuracy results with exact solution.

Step 4: Construct computer programming code

Maple 18 software is used as the platform to get the data due to its simplicity and easy to understand the execution. The result will sort into the Excel and then the graph performance profile will plot into SigmaPlot 10.0.

Step 5: Test run programming code

This step is conducted to get the correct result. If there is error in the codes of these method, Step 4 will repeat until get the fixed result. The data form from this test run is collected and tabulated into Microsoft Excel.

Step 6: Data presentation and analysis

Data collected from the results of each method are tabulated and analyzed in Microsoft Excel. Then, these data are compared by using performance profile constructed by SigmaPlot 10.0 based on the error percentage of the solution.

Step 7: This step will determine the best method found using the accuracy test.

2. IMPLEMENTATION

2.1 Theory of Galerkin Method

This illustrated for the heat conduction problems. The direct method is limited for nodal heat input. It need more advanced method for heat generation and convection heat transfer. GM can be used for this purpose to solve the problems.

Differential equation with heat generation

$$\frac{d}{dx} (kA \frac{dT}{dx}) + Q_g A = 0, 0 \leq x \leq L \tag{2.1}$$

Substitute approximate solution

$$\frac{d}{dx} (kA \frac{dT}{dx}) + Q_g A = R(x) \tag{2.2}$$



Residual

Integrate the residual with $N_i(x)$ as a weight

$$\int_{x_i}^{x_j} (\frac{d}{dx} (kA \frac{dT}{dx}) + A Q_g) N_i(x) dx = 0 \tag{2.3}$$

Integrate by parts

$$kA \frac{dT}{dx} N_i(x) \Big|_{x_i}^{x_j} - \int_{x_i}^{x_j} kA \frac{dT}{dx} \frac{dN_i}{dx} dx = - \int_{x_i}^{x_j} A Q_g N_i(x) dx \tag{2.4}$$

Substitute interpolation relation

$$\int_{x_i}^{x_j} kA (T_i \frac{dN_i}{dx} + T_j \frac{dN_j}{dx}) \frac{dN_i}{dx} dx = \int_{x_i}^{x_j} A Q_g N_i(x) dx - q(x_j) N_i(x_j) + q(x_i) N_i(x_i) \tag{2.5}$$

Perform integration

$$\frac{kA}{L^{(e)}} (T_i - T_j) = Q_i^{(e)} + q_i^{(e)}$$

$$Q_i^{(e)} = \int_{x_i}^{x_j} A Q_g N_i(x) dx$$

Repeat with $N_j(x)$ as a weight

$$\frac{kA}{L^{(e)}} (T_j - T_i) = Q_j^{(e)} + q_j^{(e)}$$

$$Q_j^{(e)} = \int_{x_i}^{x_j} A Q_g N_j(x) dx$$

Combine the two equations

$$\frac{kA}{L^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \end{Bmatrix} = \begin{Bmatrix} Q_i^{(e)} + q_i^{(e)} \\ Q_j^{(e)} + q_j^{(e)} \end{Bmatrix} \tag{2.6}$$

$$[k_T^{(e)}] \{T\} = \{Q^{(e)}\} + \{q^{(e)}\} \tag{2.7}$$

Similar to 1D bar element

$-\{Q^{(e)}\}$: thermal load corresponding to the heat source

source

$-\{q^{(e)}\}$: vector of nodal heat flows across the cross-section

Uniform heat source

$$\{Q^{(e)}\} = \int_{x_i}^{x_j} A Q_g \begin{Bmatrix} N_i(x) \\ N_j(x) \end{Bmatrix} dx = \frac{A Q_g L^{(e)}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{2.8}$$

Equally divided to the two nodes

Temperature varies linearly in element, and the heat flux is constant

2.2 Theory of Collocation Method

Assume only one PDE and one BC to solve

$$R_E(x, a) = R_E(x, a), R_B(x, a) = R_B(x, a) \tag{2.9}$$

Idea: pick n points in object (at least one in V and one on A) and require residual to be zero at each points

$$R_E(x_i, a) = 0, i = 1, 2, \dots, n_v$$

$$R_B(x_j, a) = 0, i = 1, 2, \dots, n_A$$

$$(n_v + n_A = n)$$

Weighting functions for collocation method are the Dirac delta functions:

$$w_i(x) = \delta(x - x_i), i = 1, 2, \dots, n_v$$

$$w_j(x) = \delta(x - x_j), j = 1, 2, \dots, n_A$$

Advantage is simple to formulate

Disadvantage is used mostly for problems with only one governing equation (axial, bar, beam, heat)

3. RESULT AND DISCUSSION

3.1 Introduction

This section study about the numerical solutions for both method CM and GM. The solutions with the number of nodes

$N = 6$ is used to find the better results which are stated in Chapter 3. The results are test run on by Maple 18 software through different functions from CM and GM. Then, these both methods are compared to exact solutions for different time tested. There is the formula to calculate error percentage. We also calculated the exact solution by using Maple18 code.

$$\text{error \%} = \left(\frac{|\text{Approximation} - \text{Exact}|}{\text{Exact}} \right) * 100\%$$

Maple software.

Case 1 The equation of the nonlinear PDE is given by

$$\frac{\partial^2}{\partial x^2} u(x, t) + u(x, t) - 2\left(\frac{\partial}{\partial t} u(x, t)\right) = 0$$

with boundary condition:

$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

and initial condition:

$$u(x, 0) = \sin(2\pi x), \quad 0 < x < 1$$

Case II

The equation of the nonlinear PDE is given by

$$\frac{\partial^2}{\partial x^2} u(x, t) + u(x, t) - 2\left(\frac{\partial}{\partial t} u(x, t)\right) = x$$

with boundary condition:

$$u(0, t) = 0, \quad u(1, t) = 0 \quad t > 0$$

and initial condition

$$u(x, 0) = x + 1, \quad 0 < x < 1$$

Case III

The equation of the nonlinear PDE is given by

$$\frac{\partial^2}{\partial x^2} u(x, t) + 2u(x, t) - 3x - 2\left(\frac{\partial}{\partial t} u(x, t)\right) = 0$$

with boundary condition:

$$u(0, t) = e^{-t}, \quad u(1, t) = 2e^{-t}, \quad t > 0$$

and initial condition:

$$u(x, 0) = 2x + 1 \quad 0 < x < 1$$

3.2 Result

Case I

In the table 1 below, the generated data results from CM, GM and also the exact value for each time, $t = 0.02$

Table 1: Case I

X	CM	GM	Exact	errors(C)%	errors(G)%
0	0	0.52672	0.55343	1	0.048263
0.2	0.960614	0.72113	0.704017	0.364476	0.024308
0.4	0.58369	0.383887	0.4012	0.454862	0.043152
0.6	-0.58369	-0.38389	-0.4012	0.454879	0.043141
0.8	-0.96061	-0.72113	-0.70402	0.364476	0.024308
1	0	-0.52672	-0.55343	1	0.048263

Case II

In the table 2 below, the generated data results from CM, GM and also the exact value for each time, $t = 0.02$.

Table 2: Case II

X	CM	GM	Exact	errors(C)%	errors(G)%
0	1.019227	1.13298	1.13442	0.101544	0.001269
0.2	1.2222	1.235632	1.23239	0.008268	0.002631
0.4	1.42424	1.427701	1.424438	0.000139	0.002291
0.6	1.62626	1.632903	1.626066	0.000119	0.004205
0.8	1.82835	1.824972	1.81811	0.005632	0.003774
1	2.03138	2.03138	1.916088	0.060171	0.060171

Case III

In the table 3 below, the generated data results from CM, GM and also the exact value for each time, $t = 0.02$

Table 3: Case III

X	CM	GM	Exact	errors(C)%	errors(G)%
0	0.901863	1.134421	1.13599	0.2061	0.001381
0.2	1.406599	1.432838	1.432453	0.018048	0.000269
0.4	1.82379	1.824395	1.824439	0.000356	2.42E-05
0.6	2.22999	2.226882	2.226261	0.001675	0.000279
0.8	2.67437	2.627422	2.681348	0.002602	0.020112
1	3.29496	3.027134	3.02654	0.088689	0.000196

3.2 Error analysis

From the tables above which have the three cases, the error obtained from each three cases are plotted in the graph (figure 1).

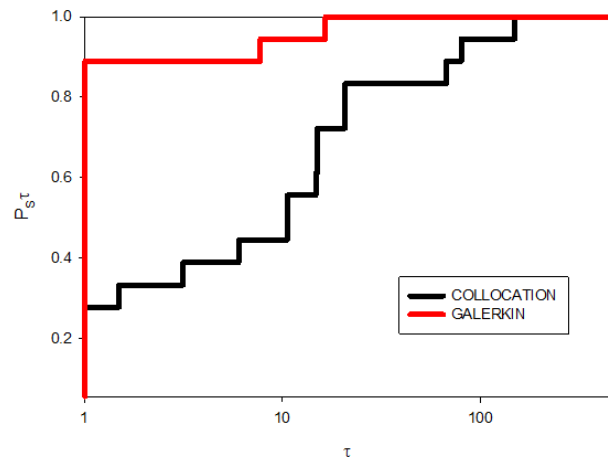


Figure 1: Performance Profile for error analysis of $t = 0.02$

5. CONCLUSION

In this study, we focused on searching the accuracy of Collocation Method (CM) and Galerkin Method (GM) in Finite Element Method (FEM) of Partial Differential Equation (PDE). Then, C-Language programming which is to be specific, Maple 18 has been used in this research in order to solve the problem.

Both method which are CM and GM will be solved particularly by using three different cases on single number of node $N = 6$ at different time. Each time will give different answer based on three cases. The different number of times are involved in calculating the results in terms of convergence and accuracy. It is important to the researcher to find the best way to solve the problem in finding the accurate result between CM and GM. From this research, increasing the number time give better result for the GM. It also showed that GM is better in terms of accuracy and convergence compared to CM.

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