ISSN 2278-3083
Volume 8, No.6, November –December 2019
International Journal of Science and Applied Information Technology
Available Online at http://www.warse.org/ijsait/static/pdf/file/ijsait07862019.pdf
https://doi.org/10.30534/ijsait/2019/07862019

h-CONVEXITY IN METRIC LINEAR SPACES

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ABSTRACT

This paper is devoted to study a new types of convex metric linear spaces .
The types of convexity considered here are a strictly h–convex, a uniformly h–convex and a locally uniformly h–convex metric linear spaces.

Key words : Strictly convex, Uniformly convex, Locally uniformly convex, s–Convex function, h–Convex function.

1. INTRODUCTION

In literature three types of convex metric linear spaces are studied, let us recall these types.
A metric linear space (X,d) is said to be :

(1) strictly convex if d (x, 0) ≤ r, d (y, 0) ≤ r imply

(2) d($\frac{x+y}{2}$,0) ≤ r unless,

uniformly convex if to each pair of positive numbers (ε, r) there corresponds a δ > 0 such that if d (x, y) ≥ ε , d (x, 0) ≤ r + δ and then d(,0) ≤

(3) locally uniformly convex if for each > 0 and x X with d (x, 0) < r δ (r > 0) , there exists a δ > 0 such that if d (x,y) ≥ and d (y, 0) < r and d(,0) ≤ r.

These types of convexities were studied in [2] and [4]. It is known that a uni- formly convex metric linear space is locally uniformly convex and a locally uniformly convex metric linear space is strictly convex. Also every totally complete strictly convex metric linear space is uniformly convex.

In the sequel of the paper, I and J are intervals in R^+ with (0, 1) ⊆ J and functions h and f are real non-negative functions defined on J and I , respectively.

Let h : J → R^+ be a non–zero, non–negative function. We say that f : I → R is an h–convex function, if f is non-negative and for all x,y ∈ I , α ∈ (0, 1) we have f (αx + (1 − α) y) ≤ h (α) f (x) + h (1 − α) f (y). This type of convexity was studied in [3].

2. MAIN RESULT

In this section, we introduce a new type of convex metric linear spaces; which so called strictly h–convex, uniformly h–convex and locally uniformly h–convex metric linear spaces and give some related results by using a technique used in [1].

Definition 2.1. Let h : J → R^+ be a non–zero, non-negative function and r ∈ (0, 1).
A metric linear space (X, d) is said to be
(1) Strictly h–convex if d (x, 0) ≤ h (r), d (y, 0) ≤ h (r) imply d($\frac{x+y}{2}$,0) ≤ h (r)
unless x = y, where x, y ∈ X and h (r) is any positive number.

(2) Uniformly h–convex if to each pair of positive numbers (ε, r) there corresponds a δ > 0 such that if x, y ∈ X are such that

$d(x, y) \geq \varepsilon$, $d(x, 0) \leq h(r) + \delta$

and

$d(x, y) \geq \varepsilon$, $d(x, 0) \leq h(r) + \delta$ and then

$$d\left(\frac{x+y}{2}, 0\right) \leq h(r) - \delta$$

(3) locally uniformly convex if for each $\varepsilon > 0$ and x ∈ X with $d(x, 0) < h(r)$ (h(r)> 0) , there exists a δ > 0 such that if $d(x,y) \geq \varepsilon$ and $d(y, 0) < h(r) + \delta$ and $d\left(\frac{x+y}{2}, 0\right) \leq h(r)$.

Also, here we note that if h (r) = r, then definition of strictly h–convex; uniformly h–convex; and locally uniformly h–convex metric linear spaces reduced to strictly convex; uniformly convex; and locally uniformly convex metric linear spaces; respectively [7].

Now, as the previous definitions of convexities of metric linear spaces we can also introduce other types of convexities of metric linear spaces related with the h–convexity above, as follows:

Let $0 < s \leq 1$, if $h(r) = r^s$ such that r ∈ (0, 1), then a metric linear space (X, d) is said to be :

(1) Strictly s–convex if $d(x, 0) \leq r^s$, $d(y, 0) \leq r^s$ imply $d\left(\frac{x+y}{2}, 0\right) \leq r^s$ unless x = y, where x, y ∈ X and r > 0 where s ∈ (0, 1] .

(2) Uniformly h–convex if to each pair of positive numbers (ε, r) there corresponds a δ > 0 such that if x, y ∈ X are such that

$d(x, y) \geq \varepsilon$, $d(x, 0) \leq r^s + \delta$

and $d(x, y) \geq \varepsilon$, $d(x, 0) \leq r^s + \delta$ and then

$$d\left(\frac{x+y}{2}, 0\right) \leq r^s - \delta$$

(3) locally uniformly convex if for each $\varepsilon > 0$ and x ∈ X with $d(x, 0) < r^s$ (r > 0) , there exists a δ > 0 such that if $d(x,y) \geq \varepsilon$ and $d(y, 0) < r^s + \delta$ and $d\left(\frac{x+y}{2}, 0\right) \leq r^s$.

Now, we give a characterization of a uniformly h–convex metric linear space, as follows.

Theorem 2. 1. Let $h : J \to R^+$ be a non–zero, non–negative function and r ∈ (0, 1).A metric linear space (X,d) is uniformly h–convex if and only if for each $\varepsilon > 0$ there exists a δ > 0 such that if for positive integers k ≥ 2, the (k + 1) elements $x_0, x_1, \ldots, x_k \in X$ are such that $d(x_i, 0) < h(r)$, (0 < r < 1,i = 0, 1, 2,...,k) and $\min_{i \neq j} d(x_i, x_j) \leq \varepsilon$, then

$$d\left(\sum_{i=0}^{k} x_i, 0\right) \leq (k+1)h(r) - \delta$$

Proof. Let $h : J \to R^+$ be a non–zero, non–negative function and r ∈ (0, 1) .Suppose first that X is uniformly h–convex. Let $x_0, x_1, \ldots, x_k$ be any (k+1) elements in X such that $d(x_i, 0) < h(r)$, (i = 0, 1, 2,...,k) and $\min_{i \neq j} d(x_i, x_j) \leq \varepsilon$ where h (r) > 0 is any positive number and k ≥ 2. Then, there exists a δ ≥ 0 such that

$$d\left(\sum_{i=0}^{k} x_i, 0\right) \leq d(x_0 + x_1, 0) + d(x_2, 0) + \ldots\ldots + d(x_k, 0)$$

$$\leq (k+1)h(r) - \delta$$

Therefore, for $\varepsilon > 0$ and h (r) > 0 , there is a

$$\delta = \left(\frac{2}{k+1}\right)\delta_1 > 0$$

such that whenever $x_0, x_1, \ldots, x_k \in X$ with $d(x_i, 0) < h(r); (0 \leq i \leq k)$ and $\min_{i \neq j} d(x_i, x_j) \geq \varepsilon$, we have

$$d(\sum_{i=0}^{k} x_i, 0) \le (k+1)h(r) - \delta$$

Conversely, suppose for a given $\varepsilon > 0$ and h (r) > 0, x,y ∈ X be such that d (x, 0) < h (r) ,

$d(y,0) < h(r)$ and $d(x,y) \ge \varepsilon$ for any $\delta > 0$ and X satisfying $(0 \le i \le k)$

Put $$x_i = (\frac{(k-i)x + iy)}{k}), (0 \le i \le k)$$

Then, x = $x_0 < x_1 < x_2 < ... < x_k$ = y are (k + 1)

elements in

hypothesis

$$d(\sum_{i=0}^{k} x_i, 0) \le (k+1)h(r) - \delta$$

This gives $$d(\frac{(k+1)(x+y)}{2}, 0) < (k+1)(h(r) - \delta).$$

Corollary 2.2. Let 0 < s ≤ 1. In Theorem 2.1. set h (r) = r^s such that r ∈(0, 1). A metric linear space (X,d) is uniformly s–convex if and only if for each $\varepsilon > 0$ there exists a δ > 0 such that if for positive integers k ≥ 2, the (k + 1) elements $x_0, x_1, ..., x_k \in X$ are such that d (x_i, 0) < r^s , (0 < r < 1, i = 0, 1, 2,...,k) and $\min_{i \ne j} d(x_i, x_j) \ge \varepsilon$, then

$$d(\frac{(k+1)(x+y)}{2}, 0) < (k+1)(r^s - \delta).$$

Proof. It's an immediate consequence of Theorem 2.1.

Now, another characterizations of uniformly h–convex metric linear space are pointed out as follows.

Theorem 2. 3. Let h : J $\to R^+$ be a non–zero, non–negative function and r ∈ (0, 1) and suppose (X,d) metric linear space , for $k \ge 1$, the following statements are are equivalent:

(1) X is uniformly h–convex.

(2) For each $\varepsilon > 0$ there exists a δ > 0 such that if {x_n} is a sequence in X with d (x_n, 0) < h (r) , (h (r) > 0, n ∈ □) and d(x_i , x_j) ≥ ε (i ≠ j). Then, there exists $\alpha_i \ge 0$ (i = 1, 2,...,k) with $\sum_{i=1}^{k} a_i^{(n)} = 1$ and

$$d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) \le h(r) - \delta, \quad (n \in □).$$

(3) For each $\varepsilon > 0$ there exists a δ > 0 such that if {x_n} is a sequence in X with d(x_n, 0) < h(r) ,(h(r) > 0, n ∈ □) and d(x_i , x_j) ≥ ε (i ≠ j). Then, for each $n \ge 1$ ther exist $a_i^{(n)} \ge 0$ with $\sum_{i=1}^{k} a_i^{(n)} = 1$ and

$$d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) \le h(r) - \delta, \quad (n \in □).$$

Proof. (1) ⇒ (2) and (2) ⇒ (3) follows in view of Theorem 2.1.

(3) ⇒ (1). Firstly we note that x_{n+i} (i = 1, 2,...,k) are k–vectors in X satisfying d (x_{n+i}, 0) < h(r), (1 ≤ i ≤ k) and d(x_{n+i},x_{n+j}) ≥ ε (i ≠ j). Therefore, by (3)

$$\begin{aligned} d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) &= d(\sum_{i=1}^{k} (1 - a_i^{(n)}) x_{n+i} + \sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0 \\ &< \text{h(r)} \sum_{i=1}^{k} (1 - a_i^{(n)}) + (\text{h(r)-}\delta) \\ &= k(\frac{\text{h(r)-}\delta}{k}) \\ &= \text{h(r)-}\delta \end{aligned}$$

Hence, by Theorem 2.1 X is uniformly h–convex.

Corollary 2.4. Let 0 < s ≤ 1. In Theorem 2.3 set h(r) = r^s such that r ∈ (0, 1) and suppose that (X,d) is a metric linear space. For k > 1, the following statements are are equivalent

(1) X is uniformly h–convex.

(2) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\{x_n\}$ is a sequence in X with $d(x_n, 0) < r^s$, $(r > 0, n \in \mathbb{N})$ and $d(x_i, x_j) \geq \varepsilon$ $(i \neq j)$. Then, there exists $\alpha_i \geq 0$ $(i = 1, 2,...,k)$ with $\sum_{i=1}^{k} a_i^{(n)} = 1$ and

$$d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) \leq r^s - \delta, \quad (n \in \mathbb{N}).$$

(3) For each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\{x_n\}$ is a sequence in X with $d(x_n, 0) < r^s$, $(r > 0, n \in \mathbb{N})$ and $d(x_i, x_j) \geq \varepsilon$ $(i \neq j)$.

Then, for each $n \geq 1$ ther exist $a_i^{(n)} \geq 0$ with $\sum_{i=1}^{k} a_i^{(n)} = 1$ and

$$d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) \leq r^s - \delta, \quad (n \in \mathbb{N}).$$

Proof. It's an immediate consequence of Theorem 2.3 .

Theorem 2. 5. Let $h : J \to R^+$ be a non–zero, non–negative function and $r \in (0, 1)$. A metric linear space (X,d) is locally uniformly h–convex if and only if for each $\varepsilon > 0$ and $x \in X$ with $d(x, 0) < h(r)$, $(h(r) > 0)$, there exists a $\delta > 0$ such that $x_0,x_1,...,x_k$ are any k elements in X satisfying $d(x_i, 0) < h(r)$, $d(x_i, x_j) \geq \varepsilon$ $(i \neq j)$ and $d(x, x_i) \geq \varepsilon$, $(i = 1, 2,...,k)$, then

$$d(\sum_{i=0}^{k} x_i, 0) \leq (k+1)(h(r) - \delta).$$

Proof. The proof can be worked out in the similar fashion.

Corollary 2.6. Let $0 < s \leq 1$. In Theorem 2.5 set $h(r) = r^s$ such that $r \in (0, 1)$, A metric linear space (X,d) is locally uniformly s–convex if and only if for each $\varepsilon > 0$ and $x \in X$ with $d(x, 0) < r^s$, $(r > 0)$, there exists a $\delta > 0$ such that $x_0,x_1,...,x_k$ are any k elements in X satisfying $d(x_i, 0) < r^s$, $d(x_i, x_j) \geq \varepsilon$ $(i \neq j)$ and $d(x, x_i) \geq \varepsilon$, $(i = 1, 2,...,k)$, then

$$d(\sum_{i=0}^{k} x_i, 0) \leq (k+1)(r^s - \delta).$$

Proof. It's an immediate consequence of Theorem 2.5.

Theorem 2. 7. Let $h : J \to R^+$ be a non–zero, non–negative function and $r \in (0, 1)$.A metric linear space (X,d) is strictly h–convex if and only if for $(k+1)$ elements $x_0,x_1,...,x_k \in X$ satisfing $d(x, 0) < h(r)$, $(h(r) > 0, \{i = 1, 2,...,k\})$, imply

$$d(\sum_{i=0}^{k} x_i, 0) \leq (k+1)h(r)$$

Unless $x_0 = x_1 = x_2 = ... = x_k$.

Proof. The proof can be worked out in the similar fashion.

Corollary 2.8. Let $0 < s \leq 1$. In Theorem 2.7 set $h(r) = r^s$ such that $r \in (0, 1)$, A metric linear space (X,d) is strictly s–convex if and only if if and only if for $(k+1)$ elements $x_0,x_1,...,x_k \in X$ satisfing $d(x, 0) < r^s$, $(r > 0, \{i = 1, 2,...,k\})$, imply

$$d(\sum_{i=0}^{k} x_i, 0) \leq (k+1)r^s$$

Unless $x_0 = x_1 = x_2 = ... = x_k$.

Proof. It's an immediate consequence of Theorem 2.7.

Theorem 2.9. For A metric space (X,d) we have,uniformly h–convexity ⇒ locally uniformly h–convexity ⇒ strictly h–convexity.

Proof. It follows from Theorems 2.1–2.3 and Theorem 2.5.

Corollary 2.10. For A metric space (X,d) we have,
uniformly s–convexity ⇒ locally uniformly s–convexity ⇒ strictly s–convexity.
Proof. It follows from Theorem 2.9

Now, we know that a metric linear space (X,d) is totally complete if its metrically bounded closed sets are compact. Therefore, we can state the following theorem.

Theorem 2.11. Every totally complete strictly h–convex metric linear space (X,d) is uniformly h–convex and thus it's locally uniformly h–convex.

Proof. Suppose that (X,d) is totally complete strictly h–convex metric linear space.

Let U ⊆ X be arbitrary compact metrically bounded closed subset of X and let {xn} be a sequence in U . Since U is bounded closed and compact subset and (X,d) is strictly h–convex then for $\varepsilon > 0$ and for a k–elements x1,...,xk in X satisfying $d(x_{n+i}, 0) < h(r)$; $(h(r) > 0$, $1 \le i \le k)$ with $d(x_{n+i}, x_{n+j}) \ge \varepsilon$ $(i \ne j)$ there exists $\delta > 0$ and for $n > 1$, $a_i^{(n)} \ge 0, (i = 1, 2, ..., k)$ with $\sum_{i=1}^{k} a_i^{(n)} = 1$, such that

$$d(\sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0) = d(\sum_{i=1}^{k}(1 - a_i^{(n)}) x_{n+i} + \sum_{i=1}^{k} a_i^{(n)} x_{n+i}, 0)$$

$$< \text{h(r)} \sum_{i=1}^{k}(1 - a_i^{(n)}) + (\text{h(r)-}\delta)$$

$$= k\,(\frac{\text{h(r)-}\delta}{k})$$

$$= \text{h(r)-}\delta$$

which means that (X,d) is uniformly h–convex and thus by Theorem 2.9. (X,d) is locally uniformly h–convex.

Corollary 2.12. Every totally complete strictly s–convex metric linear space (X, d) is uniformly s–convex and thus it's locally uniformly s–convex.
Proof. It follows from Theorem 2.11.

Remark 2.13. If s = 1 in the definitions of strictly s–convex; uniformly s–convex ; and locally uniformly s–convex metric linear space (X, d) then (X, d) reduced to strictly convex ; uniformly convex; and locally uniformly convex metric linear space ; respectively.
Remark 2.14. The above results of h–convexity holds for strictly convex ; uniformly Convex ; and locally uniformly convex metric linear space (X, d) when h(r) = r.

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